geometry of matroids - exercise session one

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some nice families of matroids

1. The matroid of the root system A_3.

   Let e_1, e_2, e_3, e_4 be the standard basis vectors of $\mathbb{R}^4$. Consider the matroid of the vector configuration
   $$\mathcal{A}_{4-1} = \{e_i - e_j \,:\, 1 \leq i < j \leq 4\}.$$
   List a few independent sets, a few dependent sets, and a few bases of this matroid.

2. Linear matroids are matroids.

   Let E be a finite set of vectors in a vector space V. Prove that the collection of linearly independent subsets of E is a matroid.

3. Graphical matroids are matroids.

   Let G be a connected graph on vertex set $\{1, \ldots, n\}$. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. Let us model the graph G geometrically, using the vector configuration
   $$V(G) = \{e_i - e_j \,:\, i < j \text{ and } ij \text{ is an edge in } G\}.$$
   Prove that a set T of edges forms a spanning tree of G if and only if the corresponding collection of vectors $V(T)$ forms a basis of $V(G)$. Conclude that graphical matroids are indeed (linear) matroids.

4. Small matroid polytopes.

   Classify all the matroid polytopes of dimension at most 3.

5. A matroid of Dyck paths.

   A *Dyck path* of length $2n$ is a path in the plane from $(0, 0)$ to $(2n, 0)$, with steps $U = (1, 1)$ and $D = (1, -1)$, that never passes below the x-axis. Each Dyck path defines an *up-step set*: the subset of $[2n]$ consisting of the positions of the steps U. For example, $UUDUDUUDDD$ is a Dyck path of length 10 and its up-step is $\{1, 2, 4, 6, 7\}$.

   Let $\mathcal{D}_n$ be the collection of up-step sets of the Dyck paths of length $2n$. Prove that $\mathcal{D}_n$ is the collection of bases of a matroid on $[2n]$.

technical exercises

6. All the bases of a matroid have the same size.

   Prove it.

7. The exchange property for bases.

   Let E be a finite set and $\mathcal{B}$ be a non-empty collection of subsets of E. Prove that $\mathcal{B}$ is the set of bases of a matroid if and only if:

   For any $A, B \in \mathcal{B}$ and $a \in A - B$, there exists $b \in B - A$ with $(A - a) \cup b \in \mathcal{B}$.

8. The symmetric exchange property for bases.

   Let $\mathcal{B}$ be the collection of bases of a matroid. Prove the *symmetric exchange property*:

   For any $A, B \in \mathcal{B}$ and $a \in A - B$, there exists $b \in B - A$ with $(A - a) \cup b,\ (B - b) \cup a \in \mathcal{B}$.