geometry of matroids

federico ardila

sorbonne université, paris
june 17 – 21, 2019

duality

1. Linear matroids as matroids of subspaces.

   Let A be a $d \times n$ matrix of rank d. Prove that the linear matroid of the columns of A equals the matroid of the subspace V=rowspace$(A) \subseteq \mathbb{R}^n$. More explicitly, for a d-subset $B \subseteq [n]$:

   $$(\text{the cols. } B \subset [n] \text{ of } A \text{ are a basis for } \mathbb{R}^d) \iff V \cap \mathbb{R}^{[n]-B} = 0 \iff V + \mathbb{R}^{[n]-B} = \mathbb{R}^n.$$

2. Matroid duality generalizes orthogonality of subspaces

   If a subspace $V \subseteq \mathbb{R}^n$ has matroid M, the orthogonal subspace $V^\perp \subseteq \mathbb{R}^n$ has matroid M^*.

3. Matroid duality generalizes duality of planar graphs

   If a connected planar graph G has matroid M, its dual graph has matroid M^*. Hints:

   (a) Let G be a connected plane graph and G^* its dual. Using the language of graph theory, describe the...

      i. bases of G.
      ii. independent sets of G.
      iii. spanning sets of G. (sets that contain a basis)
      iv. circuits of G. (minimal sets that are not independent)
      v. hyperplanes of G. (maximal sets that are not spanning)

   (b) Prove that S *is a* circuit in G if and only if $E - S$ is a hyperplane in G^*.

   (c) Conclude that S *is a* basis in G if and only if $E - S$ is a basis in G^*.

technical exercises

1. Faces of products, faces of faces.

   Let P, Q, R, F be polytopes.

   (a) If F is a face of $P \times Q$, then $F = G \times H$ for some face G of P and some face H of Q.

   (b) If R is a face of Q and Q is a face of P, then R is a face of P.

2. The matroid of bases of minimum weight. Let $M = (E, \mathcal{B})$ be a matroid and let $w : E \to \mathbb{R}$ be a weight function on E. For each real number r, let $E_r = \{e \in E \mid w(e) \leq r\}$. Notice that there are only finitely many different sets E_r; let's call them $\emptyset = S_0 \subset S_1 \subset \cdots \subset S_k \subset S_{k+1} = E$. Prove that

   $$M_w = \bigoplus_{i=0}^{k} M[S_i, S_{i+1}]$$

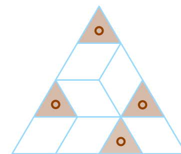
additional challenges

1. An unexpected duality.

   Why is the matroid polytope of the complete graph K_4 (or, equivalently, of the root system A_3 of problem 1 in exercise sheet 1) self-dual?

2. Matroids from tilings.

You are given an equilateral triangular board of size n divided into little unit triangles, and tiles which are little $60° - 120°$ unit rhombi. Notice that it is impossible to tile the board with the given tiles, because the board contains $\binom{n+1}{2}$ triangles facing up and $\binom{n}{2}$ facing down, and each unit rhombus must cover one triangle of each kind.



However, if you punch $\binom{n+1}{2} - \binom{n}{2} = n$ unit triangular holes, you may (or may not) be able to tile the resulting board. If it is possible to tile it, we will call the set of n holes *good*. The picture shows a good set of 4 holes, and a corresponding tiling.

Prove that the good sets of holes are the bases of a matroid.