geometry of matroids – exercise session four

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june 17 – 21, 2019

computational exercises

1. An EL-labeling.

   Let $A_3 = \{e_i - e_j \,:\, 1 \leq i < j \leq 4\}$, ordered so $12 < 13 < 14 < 23 < 24 < 34$, where ij represents $e_i - e_j$. You found the characteristic polynomial of this matroid yesterday.

   (a) Write down the EL-labeling for the lattice of flats $\mathcal{L}$ where the label of the edge from F to G is $\min(G - F)$.

   (b) For each k, count the number of decreasing chains of length k starting at the bottom of the lattice of flats $\mathcal{L}$.

   (c) For each k, count the number of decreasing chains of length k starting at the bottom of the lattice of flats $\mathcal{L}$ and avoiding the minimum label 12.

   (d) How do your answers to (b) and (c) relate to the characteristic polynomial?

2. A Bergman complex.

   Let $A_3 = \{e_i - e_j \,:\, 1 \leq i < j \leq 4\}$.

   (a) Draw the Bergman complex of M; that is, the order complex of the lattice of flats.

   (b) Verify that it is a wedge of $|\mu(M)|$ spheres of dimension $r - 2$.

   (c) Use the EL-labeling of problem 1 to identify a shelling of the Bergman complex. Identify the $|\mu(M)|$ faces that "close" the spheres during this shelling.

3. A Chow ring computation.

   Recall that the Chow ring of a matroid M is

   $$A^\bullet(M) = \mathbb{R}[x_F \,:\, F \text{ proper flat}]/(I_M + J_M)$$

   where

   $$I_M = \langle x_F x_G \,:\, F, G \text{ incomparable flats}\rangle, \qquad J_M = \langle \sum_{F \ni i} x_F - \sum_{F \ni j} x_F \;\text{ for all } i, j \in E\rangle$$

   This ring is graded of degree $r - 1$, and there is a unique isomorphism $\deg : A^{r-1}(M) \to \mathbb{R}$ such that $\deg(x_{F_1} \cdots x_{F_{r-1}}) = 1$ for any complete flag of flats $\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_{r-1} \subsetneq E$. Consider the following elements of $A^1(M)$:

   $$\alpha = \sum_{i \in F} x_F, \qquad \beta = \sum_{i \notin F} x_F.$$

   (a) Prove that α and β are well-defined, that is, they do not depend on i.

   (b) In our running example A_3, write down some expressions for α and β.

   (c) For $k = 0, 1, 2$, verify that $\alpha^{2-k}\beta^k$ is a sum of monomials corresponding to the decreasing chains of length k that start at the bottom of the lattice of flats $\mathcal{L}$ and avoid the minimum label; see 1(c). Use this to compute $\deg(\alpha^2), \deg(\alpha\beta), \deg(\beta^2)$.

   (d) How does your answer to 2(c) relate to the characteristic polynomial of M?

conceptual exercise

1. A Chow ring computation.

   Prove that $\deg(\alpha^{r-1}), \deg(\alpha^{r-2}\beta), \cdots \deg(\alpha\beta^{r-2}), \deg(\beta^{r-1})$, are the coefficients of the reduced characteristic polynomial of M for any matroid M.

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additional challenges

1. The matroid of the h-to-f transformation.

   In Martina's course, you saw how to compute the f-vector of a simplicial polytope in terms of the first half of the h-vector (since $h_i = h_{d-i}$). Explain why the matroid of this h-to-f transformation is, essentially, the same as the "Catalan matroid" of Exercise Session 1.

   (See:

   Federico Ardila, *The Catalan matroid.*

   Anastasia Chavez and Nicole Yamzon, *The Dehn-Somerville relations and the Catalan matroid.*)

2. Matroids from high-dimensional tilings.

   Let $\Delta_3 = ABCD$ be a standard tetrahedron. The *fine mixed cells* are unit tetrahedra $ABCD$, triangular prisms like $ABC \times AD$, parallelepipeds like $AB \times AC \times AD$, and parallelepipeds like $AB \times BC \times CD$ (and their permutations.) You may permute the labels A, B, C, D, but not rotate the tiles.

   (a) A *fine mixed subdivision* of the tetrahedron $n\Delta_3$ of side length n is a tiling into fine mixed cells.

      i. Prove that any such tiling must use exactly n tetrahedra.

      ii. Prove that any such tiling satisfies the "Spread Out Simplices" condition: any subtetrahedron $k\Delta_{d-1}$ of $n\Delta_{d-1}$ contains at most k of the unit tetrahedra.

      iii. (*) Suppose that you have n unit tetrahedra satisfying the Spread Out Simplices condition. Can one always find a fine mixed subdivision of $n\Delta_3$ with simplices in those positions?

   (b) i. Prove that the collections of simplices satisfying the "Spread Out Simplices" condition form a matroid.

      ii. Prove that this is the matroid of the lines of intersection of 4 generic complete flags of subspaces of $\mathbb{R}^n$.

      iii. (*) Study this matroid: how does it relate to other families of matroids?

   (c) (*) Generalize all of this to higher dimensions.

   (See:

   Federico Ardila and Sara Billey , *Flag arrangements and triangulations of products of simplices.*

   Federico Ardila and Cesar Ceballos, *Acyclic systems of permutations and fine mixed subdivisions of simplices.*

   Francisco Santos, *Some acyclic systems of permutations are not realizable by triangulations of products of simplices.*)