Algebraic and combinatorial aspects of face numbers and Stanley-Reisner rings

Exercise sheet – Day 1

Exercise 1 [The dual of a polytope]

Recall that the dual of a polytope $P \subseteq \mathbb{R}^d$ is the set $P^* := \{p \in \mathbb{R}^d : \langle x, p\rangle \leq 1 \text{ for every } x \in P\}$.

i. Prove that P^* is a polytope if and only if $0 \in \operatorname{int}(P)$.

ii. Compute the vertices of the dual of $P = \operatorname{conv}\left(\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}\right)$.

iii. Prove that the face lattice of P^* is isomorphic to the face lattice of P with all relations reversed.

Exercise 2 [Cyclic polytope]

Recall that a d-dimensional cyclic polytope $C(n, d)$ is the convex hull of n distinct points on the monoment curve $\{q_d(t) := (t, t^2, \ldots, t^d) : t \in \mathbb{R}\}$. Prove that $C(n, d)$ satisfies the following properties:

i. $\dim(C(n, d)) = d$.

ii. $C(n, d)$ is $\lfloor \frac{d}{2} \rfloor$-neighborly, i.e., $f_i(C(n, d)) = \binom{n}{k}$ for every $0 \leq k \leq \lfloor \frac{d}{2} \rfloor$.

iii. (Gale evenness criterion). For every set $V_d = \{t_{i_1}, \ldots, t_{i_d}\}$ the set $\{q_d(t_{i_1}), \ldots, q_d(t_{i_d})\}$ is the set of vertices of a facet F of $C(n, d)$ if and only if for every two points $t_i < t_j \in V \backslash V_d$ the number $|V_d \cap \{t_i, t_{i+1} \ldots, t_j\}|$ is even. Conclude that the face lattice of $C(n, d)$ does not depend on the choice of points on the moment curve.

iv. Derive a closed formula for the number $f_{d-1}(C(n, d))$.

Exercise 3 [f-vectors of 3-polytopes]

i. Show that the set of f-vectors of 3-polytopes P is given by $C \cap \mathbb{Z}^3$, with $C \subseteq \mathbb{R}^2$ a 2-dimensional convex cone.
Hint: Reduce the problem to the study of pairs $(f_0(P), f_2(P))$.

ii. Compute $\frac{1}{2}(f(C(5, 4)) + f(C(9, 4)))$. Conclude that a description for the set of f-vectors of 4-polytopes as in the 3-dimensional case is not possible.

Exercise 4 [Complicated numbers but simple polytopes]

Let $P \subseteq \mathbb{R}^d$ be a simple polytope and $\ell : \mathbb{R}^d \to \mathbb{R}$ be a generic linear functional that is injective on the vertices. Let $h_i^\ell(P)$ be the number of vertices of indegree i in the graph of P oriented in a way such that $u \to v$ if and only if $\ell(u) < \ell(v)$.

i. Let $v_0 \in P$ be a fixed vertex. Prove that if $\ell(u) < \ell(v_0)$ for every edge $\{u, v_0\} \in P$, then $\ell(u) < \ell(v_0)$ for every vertex u other than v_0.

ii. Use i. to conclude that

$$\sum_{k=0}^{d} f_k(P)x^k = \sum_{i=0}^{d} h_i^\ell(P)(x+1)^i.$$

In particular $h^\ell(P)$ does not depend on ℓ and we can define $h(P) = h^\ell(P)$ for some generic linear functional ℓ.

iii. Compute $h_0(P)$, $h_1(P)$ and $h_d(P)$ as functions of the f-vector of P.

Exercise 5 [The h-numbers of a facet]

Let $P \subseteq \mathbb{R}^d$ be a simple polytope and F be a facet of P.

i. Show that $h_i(P) \geq h_{i-1}(F)$ for every $i = 1, \ldots, d-1$.

ii. Show that

$$\sum_{F \text{ facet of } P} h_i(F) = (i+1)h_{i+1}(P) + (d-i)h_i(P),$$

for every $i = 0, \ldots, d-1$.

Hint: It is convenient to fix an orientation of the graph of P induced by a generic linear functional as in the lecture. Then $h_i(P) = h_i^\ell(P) = |\{v : \text{in-deg}(v) = i\}$.

Exercise 6 [h-vectors of V.I.P.s, Very-Important-Polytopes]

Compute the numbers $h_i(P)$ when:

i. $P = [-1, 1]^d$, i.e., P is the d-dimensional *cube.*

ii. $P = \text{conv}(\pm e_1, \ldots, \pm e_d)$, with $e_1, \ldots, e_d$, i.e., P is the d-dimensional *cross-polytope.*

iii. $P = C(n, d)$, i.e., P is a d-dimensional cyclic polytope on n vertices.

iv. $P = C(n, d)^*$.

v. $P = \text{conv}(\{(\pi(1), \ldots, \pi(d)) : \pi \text{ is a permutation on } [d]\})$, i.e., P is the $(d-1)$-dimensional *permutahedron.*

vi. $P = \Delta_{i_1} \times \cdots \times \Delta_{i_k}$ for $0 \leq i_1 \leq \cdots \leq i_k$ and Δ_j the j-dimensional simplex.