Geometric and combinatorial aspects of face numbers 6/24/19

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Day 1 — Infinitesimal Rigidity

A short review:

Let $G = (V, E)$ be a framework in $\mathbb{R}^d$, where $V = \{v_1, \dots, v_n\}$. Recall that an *infinitesimal motion of* G is an assignment of vectors $u_1, \dots, u_n \in \mathbb{R}^d$ such that

$$\langle v_i - v_j, u_i - u_j \rangle = 0 \qquad \forall ij \in E.$$

The set of all infinitesimal motions of $\mathbb{R}^d$ is a vector space denoted $M(G)$. Similarly, an *infinitesimal motion of* $\mathbb{R}^d$ is a function $u : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\langle v - \overline{v}, u(v) - u(\overline{v}) \rangle = 0 \quad \forall v, \overline{v} \in \mathbb{R}^d,$$

and the space of infinitesimal motions of $\mathbb{R}^d$ is denoted by $M(\mathbb{R}^d)$.

An infinitesimal motion $u_1, \dots, u_n$ of G is called *trivial* if there is some infinitesimal motion u of $\mathbb{R}^d$ such that $u_i = u(v_i)$ for all $i \in [n]$. A framework G is *infinitesimally rigid* if it admits only trivial infinitesimal motions.

A new definition: The *stress space of* G, $S(G)$, is defined as the kernel of the rigidity matrix $R(G)$. In other words, a stress $\lambda = (\lambda_{ij})$ on G is an assignment of numbers λ_{ij} to edges $ij \in E$ such that the following equilibrium conditions hold:

$$\sum_{j : ij \in E} \lambda_{ij}(v_j - v_i) = 0 \quad \forall i \in [n].$$

Exercises

We will now prove several basic results about infinitesimal rigidity and stress spaces.

0. As a warm-up, check that $u_1, \dots, u_n$ is an infinitesimal motion of G if and only if,

$$\text{proj}_{v_i - v_j}(u_i) = \text{proj}_{v_i - v_j}(u_j) \quad \forall ij \in E,$$

where $\text{proj}_v(u)$ is the vector projection of u along v.

1. The goal of this problem is to show that $\dim M(\mathbb{R}^d) = \binom{d+1}{2}$.

   (a) Let $v_1, \dots, v_d, v_{d+1}$ be the vertices of a geometric d-dimensional simplex Σ in $\mathbb{R}^d$ and let $G(\Sigma)$ be the framework corresponding to the graph of Σ (in particular, it is a complete graph). Let $u_1, \dots, u_d, u_{d+1}$ be an infinitesimal motion of $G(\Sigma)$. Show that u_{d+1} is determined by $u_1, \dots, u_d$.

   (b) Use part (a) to prove that $\dim M(G(\Sigma)) = \binom{d+1}{2}$.

(c) Show that distinct infinitesimal motions of $\mathbb{R}^d$ induce distinct infinitesimal motions of $G(\Sigma)$, and conclude that $\dim M(\mathbb{R}^d) \leq \dim M(G(\Sigma)) = \binom{d+1}{2}$.

(d) Compute the dimension of the space of isometries of $\mathbb{R}^d$ (i.e., translations and rotations). Use this fact to show that $\dim M(\mathbb{R}^d) \geq \binom{d+1}{2}$. The promised result follows.

2. Let $G = (V, E)$ be a framework in $\mathbb{R}^d$ and assume that V affinely spans $\mathbb{R}^d$. Show that the result of Problem 1 implies the following facts:

   (a) G is infinitesimally rigid if and only if $\dim M(G) = \binom{d+1}{2}$. In this case, every infinitesimal motion of G is determined by its restriction to any d affinely independent joints of G.

   (b) G is infinitesimally rigid if and only if the rank of the rigidity matrix of G, $\mathcal{R}(G, d)$, is $df_0 - \binom{d+1}{2}$.

   (c) G is infinitesimally rigid if and only if $\dim S(G) = f_1 - df_0 + \binom{d+1}{2}$.

   Here f_0 denotes the number of vertices of G while f_1 denotes the number of edges of G.

3. Use Problem 2 to prove the following results.

   (a) The *Gluing Lemma*: If $G = (V, E)$ and $G' = (V', E')$ are two d-dimensional frameworks that have d affinely independent joints in common, then the d-dimensional framework $G \cup G' = (V \cup V', E \cup E')$ is also infinitesimally rigid.

   (b) The *Cone Lemma*: Let $G = (V, E)$ be a framework in $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$ and let $u \in \mathbb{R}^{d+1} \setminus \mathbb{R}^d \times \{0\}$ define $G * u := (V', E')$ to be the cone over G with apex u, that is, $V' = V \cup \{u\}$ and $E' = E \cup \{uv : v \in V\}$. Show that G is infinitesimally rigid in $\mathbb{R}^d$ if and only if $G * u$ is infinitesimally rigid in R^{d+1}.
   Hint: what is the relationship between the stress spaces of G and $G * u$?

4. Let $G = ([n], E)$ be a graph on n vertices (considered as an abstract graph). A map $\psi : [n] \to \mathbb{R}^d$ is called a *d-embedding* of G. We identify the set of all d-embeddings of G with $\mathbb{R}^{dn}$ via $\psi \mapsto (v_1, \ldots, v_n) = (\psi(1), \ldots, \psi(n)) \in \mathbb{R}^d \times \ldots \times \mathbb{R}^d \cong \mathbb{R}^{dn}$. Each d-embedding ψ of G defines a d-dimensional framework whose joints are $\psi(1), \ldots, \psi(n)$ and whose bars are the elements of E. A d-embedding of G into $\mathbb{R}^d$ is *infinitesimally rigid (flexible,* resp.) if the corresponding framework is infinitesimally rigid (flexible, resp.).

   (a) Prove that if the set of infinitesimally rigid d-embeddings of a graph with n vertices is *not empty*, then it is open and dense in $\mathbb{R}^{dn}$. For this reason, a graph that possesses an infinitesimally rigid embedding in $\mathbb{R}^d$ is called *generically d-rigid*.

   (b) (Gluing Lemma for generic rigidity) Show that if G and G' are generically d-rigid graphs that share at least d vertices, then $G \cup G'$ is a generically d-rigid graph.

   (c) (Cone Lemma for generic rigidity) Show that a graph G is generically d-rigid if and only if the cone over G is generically $(d + 1)$-rigid.

   Hint for part (a): Show that the set of infinitesimally flexible d-embeddings of G is an algebraic variety, and hence its complement in $\mathbb{R}^{dn}$ is either empty or is open and dense.