Geometric and combinatorial aspects of face numbers 6/25/19

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Day 2 — μ-numbers and the fundamental group

A few definitions:

Let Γ and Δ be pure simplicial complexes of the same dimension on disjoint vertex sets. Let F and G be facets of Γ and Δ respectively, and let $\varphi : F \to G$ be a bijection between the vertices of F and the vertices of G. The **connected sum** of Γ and Δ, denoted $\Gamma \#_{\varphi} \Delta$ or simply $\Gamma \# \Delta$, is the simplicial complex obtained by identifying the vertices of F and G (and all faces on those vertices) according to the bijection φ and removing the facet corresponding to F (which has been identified with G).

Let Δ be a pure simplicial complex of dimension $d-1$, and let F and F' be facets of Δ with disjoint vertex sets. If there is a bijection $\varphi : F \to F'$ such that v and $\varphi(v)$ do not have a common neighbor in Δ for every $v \in F$, the simplicial complex Δ^{φ} obtained from Δ by identifying the vertices of F and F' (and all faces on those vertices) and removing the facet corresponding to F (which has been identified with F') is called a **handle addition** to Δ. The requirement that v and $\varphi(v)$ do not have a common neighbor in Δ ensures that Δ^{φ} is a simplicial complex.

Exercises

1. Today we discussed the following theorem: if Δ is a connected simplicial manifold of dimension $d-1 \geq 3$, then $h_2(\Delta) - h_1(\Delta) \geq \binom{d+1}{2} m(\Delta)$, where $m(\Delta)$ is the minimum number of generators of $\pi_1(\Delta)$. The goal of this exercise is to show that this bound is sharp.
   - (a) Let Δ_1 and Δ_2 be two pure simplicial complexes of dimension $d-1 \geq 3$. Express h_1- and h_2-numbers of $\Delta_1 \# \Delta_2$ in terms of the h-numbers of Δ_1 and Δ_2.
   - (b) Let Δ be a pure simplicial complex of dimension $d-1$, and let Δ^{φ} be obtained from Δ by handle addition. Express the h_1- and h_2-numbers of Δ^{φ} in terms of $h(\Delta)$.
   - (c) It is known that for $d \geq 4$, the operation of handle addition increases the value of the m-number by 1. Use this fact and your results in the previous parts to show that for every non-negative integer b and $d \geq 4$, there exists a connected simplicial manifold Δ of dimension $d-1$ such that $m(\Delta) = b$ and $h_2(\Delta) = h_1(\Delta) + \binom{d+1}{2} b$.
2. The goal of this exercise is to prove the Morse-type inequalities on the μ-numbers. Let Δ be a simplicial complex and let $\varsigma = (v_1, v_2, \ldots, v_n)$ be a linear ordering of its vertices.
   - (a) Note that for $k \leq n$, $\Delta_{\{v_1,\ldots,v_k\}} = \Delta_{\{v_1,\ldots,v_{k-1}\}} \cup \mathrm{st}(v_k, \Delta_{\{v_1,\ldots,v_k\}})$. Apply the Mayer-Vietoris sequence to show that for all $i \geq 0$,
   $$\tilde{\beta}_i(\Delta_{\{v_1,\ldots,v_k\}}) \leq \tilde{\beta}_i(\Delta_{\{v_1,\ldots,v_{k-1}\}}) + \tilde{\beta}_{i-1}(\mathrm{lk}(v_k, \Delta_{\{v_1,\ldots,v_k\}}).$$
   - (b) Use part (a) to conclude that for all $i \geq 0$, $\tilde{\beta}_i(\Delta) \leq \mu_i^{\varsigma}(\Delta)$ for any ordering ς of the vertex set of Δ. Then conclude that $\tilde{\beta}_i(\Delta) \leq \mu_i(\Delta)$ for all $i \geq 0$.

(c) Can you extend this argument to show that $\sum_{j=0}^{i}(-1)^{i-j}\tilde{\beta}_j(\Delta) \leq \sum_{j=0}^{i}(-1)^{i-j}\mu_j(\Delta) + (-1)^{i+1}$?

3. This exercise along with some careful bookkeeping can be used to show that for any simplicial complex Δ, $\mu_1(\Delta) - \mu_0(\Delta) + 1 \geq m(\Delta) - \tilde{\beta}_0(\Delta)$.

   (a) Let Δ and Γ be two simplicial complexes. Assume that Δ is connected, Γ is contractible, and that $\Delta \cap \Gamma$ has exactly two connected components. Prove that $m(\Delta \cup \Gamma) \leq m(\Delta) + 1$. **Hint**: Let Σ_1 and Σ_2 be the two connected components of $\Delta \cap \Gamma$, and let γ be the shortest path in Δ that starts in Σ_1 and ends in Σ_2. Consider $\Gamma' = \Gamma \cup \gamma$. Apply the Seifert-van Kampen theorem to $\Delta \cup \Gamma = \Delta \cup \Gamma'$.

   (b) Prove that if Δ is a connected simplicial complex, $\Gamma_1, \ldots, \Gamma_s$ are several connected and pairwise disjoint subcomplexes of Δ, and v is vertex not in Δ, then

$$m\left(\Delta \cup \left(v * \left(\Gamma_1 \cup \cdots \cup \Gamma_s\right)\right)\right) \leq m(\Delta) + s - 1.$$

4. Let Δ be a simplicial complex with vertex set V, and let S_V be the set of all linear orderings of V. Recall that for $\varsigma = (v_1, \ldots, v_n) \in S_V$,

$$\mu_j^{\varsigma}(\Delta) := \sum_{k=1}^{n} \tilde{\beta}_{j-1}\left(\operatorname{lk}(v_k, \Delta_{\{v_1,\ldots,v_k\}})\right), \text{ and that } \mu_j(\Delta) := \frac{1}{|V|!}\sum_{\varsigma \in S_V} \mu_j^{\varsigma}(\Delta).$$

For a simplicial complex Γ on vertex set U, we also define

$$\sigma_j(\Gamma) := \sum_{W \subseteq U} \frac{1}{\binom{|U|}{|W|}} \tilde{\beta}_j(\Gamma_W) \quad \text{for all } -1 \leq j \leq \dim \Gamma.$$

Prove that

$$\mu_j(\Delta) = \sum_{v \in V} \frac{\sigma_{j-1}(\operatorname{lk}(v, \Delta))}{f_0(\operatorname{lk}(v, \Delta)) + 1} \quad \text{for all } 0 \leq j \leq \dim \Delta. \tag{1}$$

Hint: For a given $v \in V$ and $W \subseteq V \backslash \{v\}$, in how many orderings of V will the set W show up as an initial segment followed up by v?
Remark: It follows from Hochster's formula, that for a simplicial complex Γ with n vertices, $\sigma_{j-1}(\Gamma) = \sum_{k=j}^{n} \frac{\beta_{k-j,k}(\mathbb{R}[\Gamma])}{\binom{n}{k}}$, where $\beta_{k-j,k}(\mathbb{R}[\Gamma])$ are the algebraic graded Betti numbers. Thus eq. (1) provides us with an algebraic interpretation of the μ-numbers.

5. (a) Let Γ be a $(d-1)$-dimensional homology sphere. Use Alexander duality to show that the σ-numbers defined in the previous problem are symmetric, that is, $\sigma_{j-1}(\Gamma) = \sigma_{d-1-j}(\Gamma)$ for all $0 \leq j \leq d$.

   (b) Conclude that if Γ is a d-dimensional homology manifold, then $\mu_i(\Delta) = \mu_{d-i}(\Delta)$ for all $0 \leq j \leq d$.