Geometric and combinatorial aspects of face numbers 6/27/19

Isabella Novik and Hailun Zheng

Day 3 — Extensions of Dehn–Sommerville relations, UBT, etc.

A few definitions:

The *reduced Euler characteristic* of a simplicial complex Δ is $\tilde{\chi}(\Delta) := \sum_{i=-1}^{\dim\Delta}(-1)^i f_i(\Delta)$. By the Euler–Poincaré theorem, the reduced Euler characteristic is also the alternating sum of the reduced Betti numbers: $\tilde{\chi}(\Delta) = \sum_{i=0}^{\dim\Delta}(-1)^i\tilde{\beta}_i(\Delta)$.

A pure simplicial complex Δ of dimension $d-1$ is called *Eulerian* if for all $F \in \Delta$, $\tilde{\chi}(\mathrm{lk}(F,\Delta)) = \tilde{\chi}(S^{d-|F|-1}) = (-1)^{d-|F|-1}$. (Here S^k is a k-dimensional sphere). A pure simplicial complex is called *semi-Eulerian* if all its vertex links are Eulerian. For instance, all simplicial spheres are Eulerian while all simplicial manifolds are semi-Eulerian.

Exercises

1. The goal of this exercise is to prove the following version of Dehn-Sommerville equations (it is due to Victor Klee, 1964): Let Δ be a semi-Eulerian simplicial complex of dimension $d-1$. Then
$$h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j\binom{d}{j}\left((-1)^{d-1}\tilde{\chi}(\Delta)-1\right).$$
   (a) Use double counting to show that for all $0 \le i \le j \le d-1$,
$$\binom{j}{i} f_{j-1}(\Delta) = \sum_{G\in\Delta,\ |G|=i} f_{j-i-1}(\mathrm{lk}(G,\Delta)).$$
   (b) Use part (a) and the assumption that Δ is semi-Eulerian to show that
$$\sum_{j=i}^{d}(-1)^{d-j}\binom{j}{i} f_{j-1}(\Delta) = f_{i-1}(\Delta) \quad \text{for all } i = 1,2,\ldots,d.$$
   What other piece of information do we know that can serve as the equation corresponding to the $i=0$ case?

   (c) Mulitply the i-th equation in part (b) by $(x-1)^{d-i}$ and sum over all $i = 0,1,\ldots,d$. Compare the coefficients of the resulting polynomials to derive the Dehn-Sommerville relations.

2. Use Problem 1 to prove the following statements.

   (a) If Δ is a $(d-1)$-dimensional Eulerian complex, then $h_i(\Delta) = h_{d-i}(\Delta)$ for all $0 \le i \le d$.

   (b) An odd-dimensional semi-Eulerian simplicial complex is Eulerian.

(c) Let Δ be a connected orientable simplicial manifold of dimension $d-1$. Show that $h''_i(\Delta) = h''_{d-i}(\Delta)$ for all $0 \le i \le d$.
Hint: Recall that by Poincaré duality $\tilde{\beta}_i(\Delta) = \tilde{\beta}_{d-i-1}(\Delta)$ for $1 \le i \le d-2$, while $\tilde{\beta}_0(\Delta) = 0$ and $\tilde{\beta}_{d-1}(\Delta) = 1$.

3. The goal of this problem is to prove the Upper Bound Theorem for odd-dimensional simplicial manifolds.

(a) Let Δ be a pure simplicial complex of dimension $d-1$. Define the *short simplicial h-numbers* of Δ by $\widetilde{h}_i(\Delta) := \sum_{v \in V(\Delta)} h_i(\mathrm{lk}(v, \Delta))$ for all $0 \le i \le d-1$. Show that if Δ is a semi-Eulerian complex, then $\widetilde{h}_i(\Delta) = \widetilde{h}_{d-i-1}(\Delta)$ for all $0 \le i \le d-1$.

(b) Show that if Δ is a pure simplicial complex, then the f-numbers of Δ are non-negative linear combinations of the $\widetilde{h}$-numbers of Δ.

(c) Compute the $\widetilde{h}$-vector of the boundary complex of the $2k$-dimensional cyclic polytope with n vertices, $C(2k, n)$.

(d) Use Stanley's UBT for spheres and previous parts of this problem to show that if Δ is a $(2k-1)$-dimensional simplicial manifold with n vertices, then $f_j(\Delta) \le f_j(C(2k, n))$ for all $1 \le j \le 2k-1$.

4. Let Δ be a $2k$-dimensional simplicial sphere. Define yet another modification of the h-numbers: $\hat{h}_i(\Delta) := \sum_{j=0}^{i} h_j(\Delta)$ for $i = 0, 1, \ldots, k$.

(a) Show that $f_{2k}(\Delta) = 2\hat{h}_k(\Delta)$; compute the $\hat{h}$-numbers of the boundary complex of $C(2k+1, n)$.

(b) Show that $(\hat{h}_0(\Delta), \hat{h}_1(\Delta), \ldots, \hat{h}_k(\Delta))$ is an M-sequence.

(c) Show by (reverse) induction that if $f_{2k}(\Delta) \ge f_{2k}(C(2k+1, n))$, then $\hat{h}_i(\Delta) \ge \hat{h}_i(C(2k+1, n))$ for all $0 \le i \le k$.

(d) Use part (c) to derive the following form of the Strong Upper Bound Theorem: if Δ is a $2k$-dimensional simplicial sphere with $f_{2k}(\Delta) \ge f_{2k}(C(2k+1, n))$, then $f_i(\Delta) \ge f_i(C(2k+1, n))$ for all $0 \le i \le 2k$.

Note: the usual UBT implies that if $f_{2k}(\Delta) \ge f_{2k}(C(2k+1, n))$, then $f_0(\Delta) \ge n$.

5. Let $\Gamma \subset \Delta$ be Buchsbaum simplicial complexes, both of dimension $d-1$. Let $\theta_1, \ldots, \theta_d$ be an l.s.o.p. for $\mathbb{R}[\Delta]$.

(a) Use the Kind-Kleinschmidt criterion to show that the natural images $\bar{\theta}_1, \ldots, \bar{\theta}_d$ of $\theta_1, \ldots, \theta_d$ in $\mathbb{R}[\Gamma]$ form an l.s.o.p. for $\mathbb{R}[\Gamma]$.

(b) Show that there is a surjection from $\mathbb{R}[\Delta]/(\theta_1, \ldots, \theta_d)$ to $\mathbb{R}[\Gamma]/(\bar{\theta}_1, \ldots, \bar{\theta}_d)$.

(c) Conclude that $h'_i(\Gamma) \le h'_i(\Delta)$ for all $0 \le i \le d$.