Polyhedral Subdivisions in Non-Parametric Statistics

Exercises for the Course by **Bernd Sturmfels** and **Kaie Kubjas** in Paris, June 24-28, 2019

Monday, June 24: Regular Subdivisions and Secondary Polytopes

(1) Compute the f-vector of the 5-dimensional associahedron. Up to symmetry, how many subdivisions does a regular octagon have? List the GKZ vectors of all triangulations.

(2) Prove the following fact: For fixed a sample $X = \{x_1, \ldots, x_n\}$ in $\mathbb{R}^d$, the set of all tight vectors y is a convex cone in $\mathbb{R}^n$. How to find linear inequalities that define this cone?

(3) Show that every triangulation of the regular 3-cube is regular. How about the 4-cube?

(4) Fix $d = 3$ and $n = 7$. Draw a picture of the secondary polytope $\Sigma(X)$ for the sample

$$X \;=\; \big\{(2,1,0),(0,2,1),(1,0,2),(1,1,1),(1,1,0),(1,0,1),(0,1,1)\big\}.$$

(5) Multiply all $1+9+9 = 19$ minors of a 3×3-matrix of unknowns (x_{ij}). List all vertices of the Newton polytope of this polynomial. This is the secondary polytope of ...... ?

(6) Fix $d = 1$, $n = 3$ and $X = \{2,5,7\}$. The Samworth body $\mathcal{S}(X)$ lives in $\mathbb{R}^3$. Draw it.

Tuesday, June 25: Geometry of Log-Concave Density Estimation

(7) Prove that the pdf of every Gaussian distribution is log-concave.

(8) Let $X = \{2,5,7\}$ as in Problem (6) and consider the function $\exp(h_{X,y})$. For which vectors $y = (y_1, y_2, y_3)$ is this a probability density function. Can you draw a picture?

(9) Let $d = 3, n = 7$ and X as in Problem (4). The integral $\int_P \exp(h_{X,y}(t))dt$ is a piecewise rational function in $y = (y_1, \ldots, y_7)$. Write an explicit rational formula for each piece.

(10) Fix a triangulation Δ of X in the previous exercise. The vector w^Δ is a function of y. Write its formula explicitly. What does this vector mean for the Samworth body $\mathcal{S}(X)$?

(11) Prove that the symmetric function $H : \mathbb{R}^d \to \mathbb{R}$ is positive, increasing and convex.

(12) Fix X and Δ as in Problem (10) and consider $y = (c,c,c,c,c,c,c)$. For which $c \in \mathbb{R}$ is $\exp(h_{X,y})$ a probability density function? Show that the vector w^Δ is a GKZ vector.

(13) On November 8, 2018, Brian Axelrod and Gregory Valiant from Stanford posted an article on the `arXiv` in the category `cs.DS`. What is the main idea in their article?

Thursday, June 27: Getting your Hands Dirty

(14) Why is the software package used by statisticians called `R`? Install it on your laptop.

(15) What is `LogConcDEAD`? Can you install this as well and get it to run it?

(16) Compute the log-concave MLE for the sample X in (4). Repeat with random weights.

(17) Let X be a sample of 500 points from the normal distribution with `set.seed(222)`. In your opinion, how many cells does the support of the log-concave MLE have?

(18) Fix the following sample of 14 points in $\mathbb{R}^2$.

$$X = \{(9,9),(3,3),(0,9),(0,1),(6,3),(1,4),(7,8),\\ (2,4),(8,9),(7,6),(6,9),(9,5),(2,6),(5,5)\}.$$

In your opinion, how many cells does the support of the log-concave MLE for X have?

(19) In which cell does the mean of the log-concave MLE lie in Problems (17) and (18)?

(20) Triangulate the 3-cube into five tetrahedra. Find a weight vector $w \in \mathbb{R}^8$ such that the log-concave MLE for $X = \{0,1\}^3$ with weights w is supported on your triangulation.

Friday, June 28: Being Totally Positive

(21) Which Gaussian distributions are log-supermodular?

(22) Compute the mix-max closure $\overline{X}$ for the set X in Problem (4). Repeat ten times for random sample of 7 points in $\mathbb{R}^3$. What is the largest observed cardinality of the set $\overline{X}$?

(23) Continue the previous experiment and compute $\mathrm{MMconv}(X)$ for various samples X in $\mathbb{R}^3$. What is the largest number of vertices or facets of such a bimonotone 3-polytope?

(24) Prove the 2-D Projections Theorem.

(25) Prove that the set $\{0,1\}^n$ is tidy. Which subsets of $\{0,1\}^n$ are tidy?

(26) What is the *Lovasz extension* of a submodular function? Why is it important?

(27) Is the set X in Problem (4) tidy or not? Which functions $h_{X,y}$ are supermodular?