Wide hollow polytopes

Giulia Codenotti

joint work with Francisco Santos

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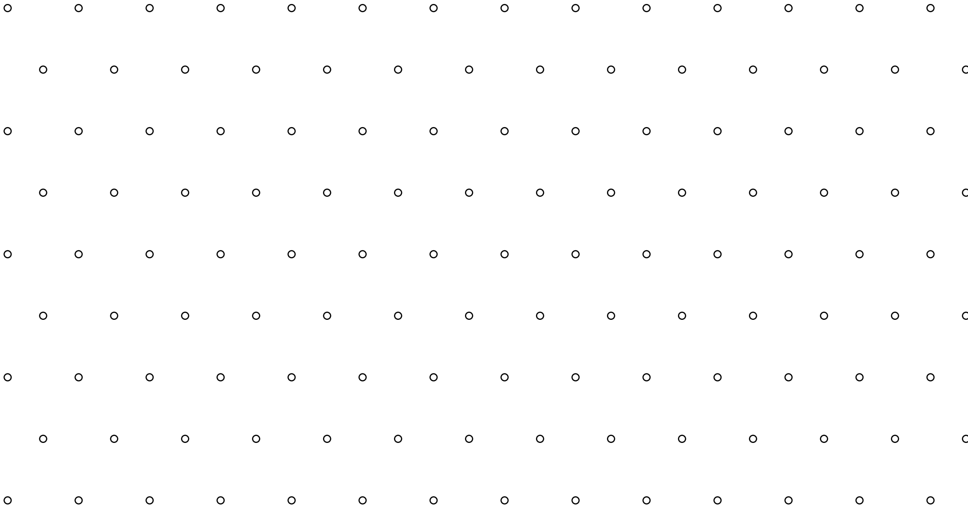
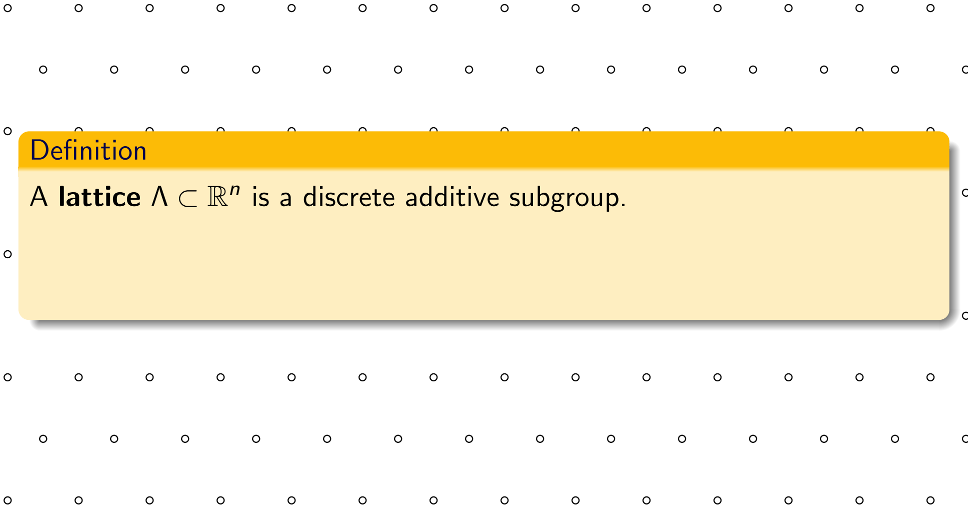
GAC summer school

Lattices





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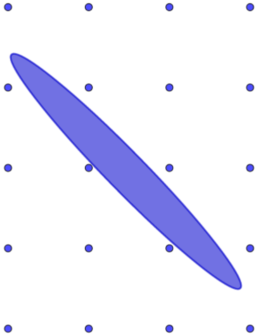
The **dual lattice** $\Lambda^* \subseteq (\mathbb{R}^n)^*$ of Λ is the lattice of functionals taking integer values on points of Λ.

Convex bodies and lattice polytopes

A **convex body**:



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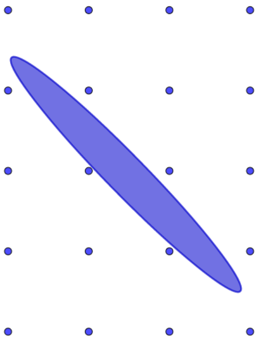

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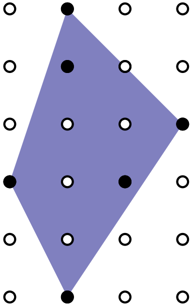

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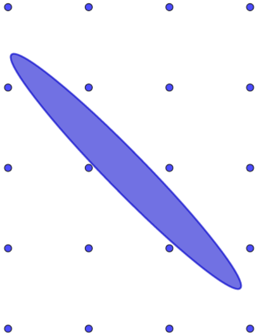

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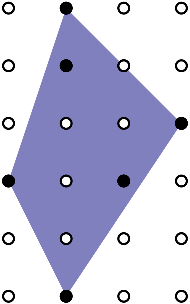

A convex body or lattice polytope is **hollow** (or lattice-free) if there are no lattice points in its interior.

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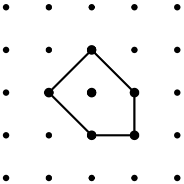

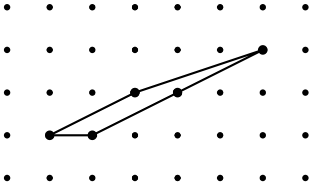

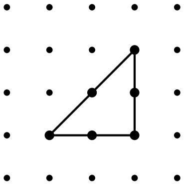

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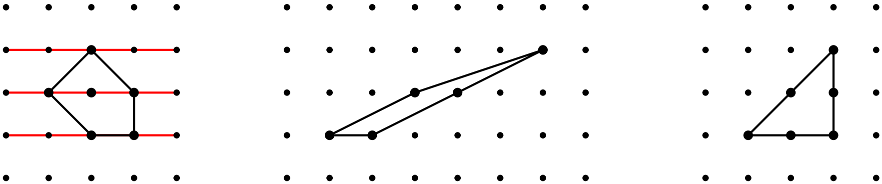

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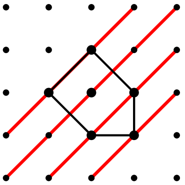

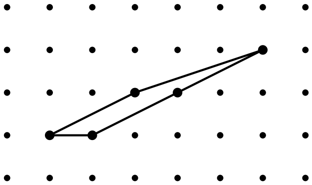

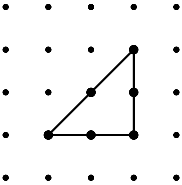

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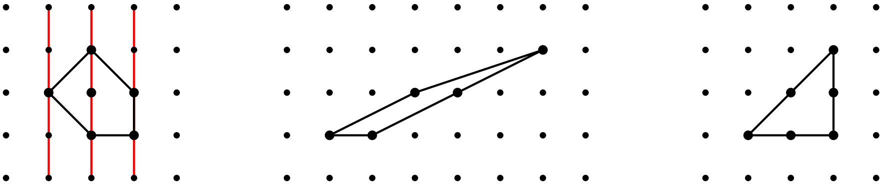

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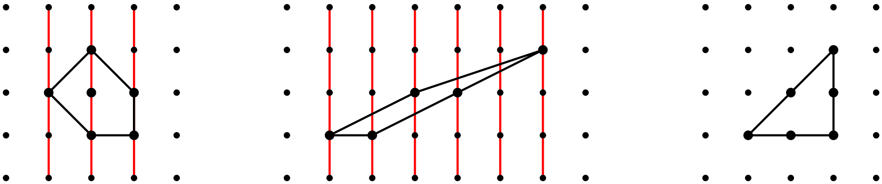

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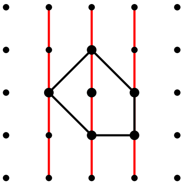

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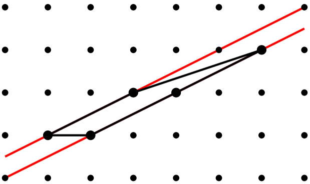

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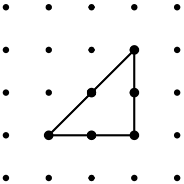

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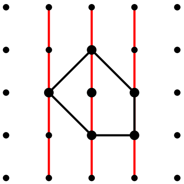

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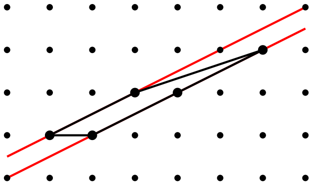

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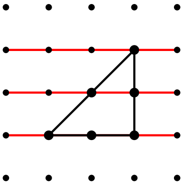

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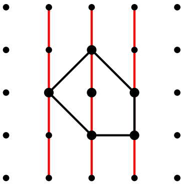

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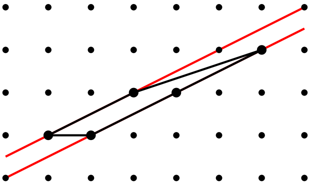

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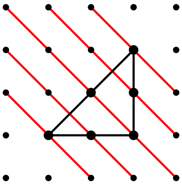

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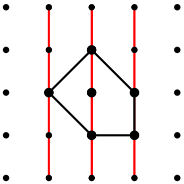

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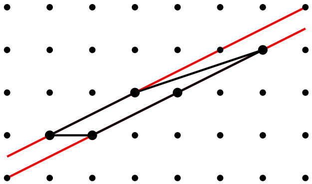

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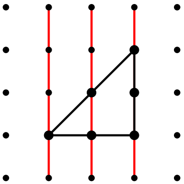

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Our goal: improve lower bounds on the flatness constant, that is, **construct hollow convex bodies/polytopes of large width**.

Variations on flatness constants

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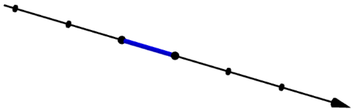

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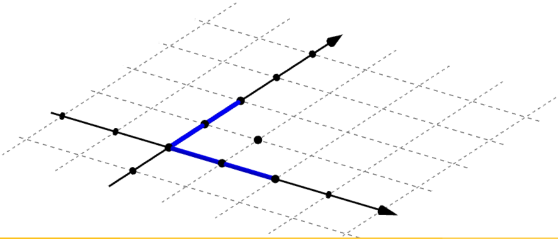

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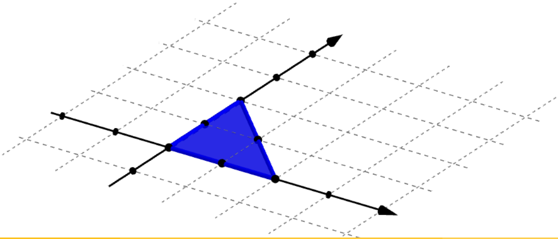

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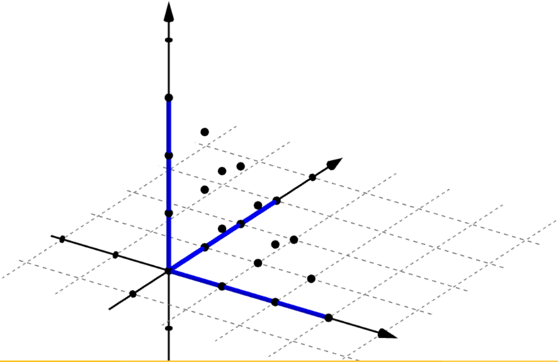

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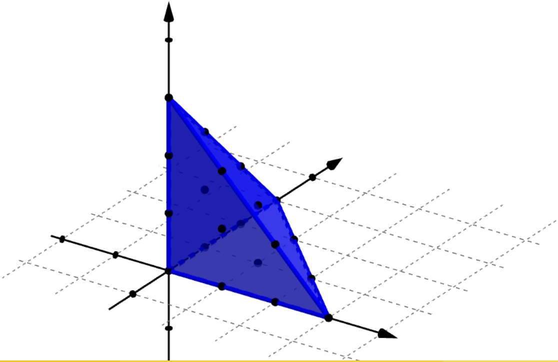

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What can we say in low dimension?

d	$w_s(d)$	$w_p(d)$	$w_c(d)$
1	1	1	1
2	2	2	$1+\frac{2}{\sqrt{3}}$ [Hur90]

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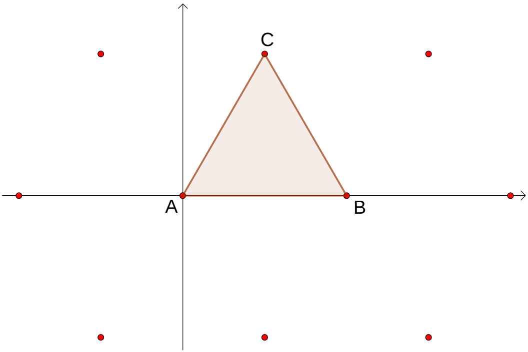
4	??		??		??	
14			≥ 15	[C-S18]		
404	≥ 408	[C-S18]				

$w_c(2)$: Hurkens' construction



A triangular lattice and a unimodular triangle ABC.

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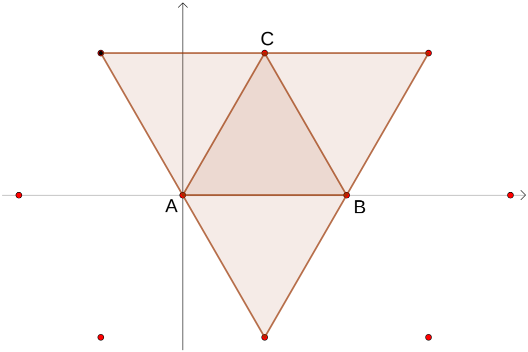

Lattice triangle circumscribed around ABC; only lattice triangle of width 2.

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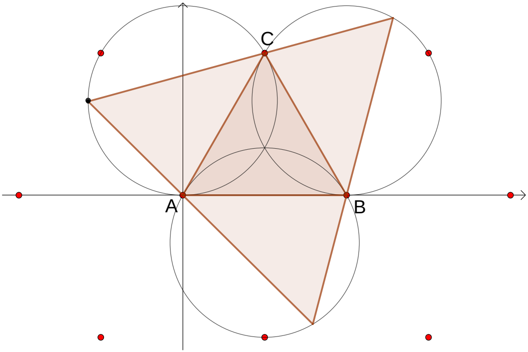

This triangle, also circumscribed around ABC, has lattice width $1 + \frac{2}{\sqrt{3}}$.

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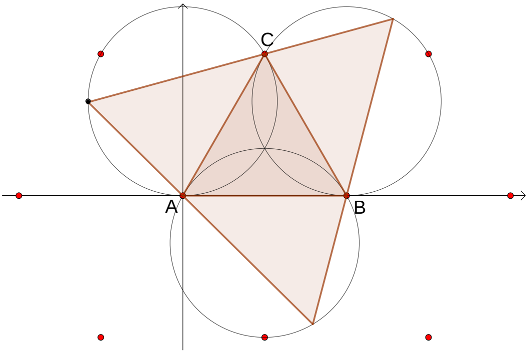

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Theorem (Hurkens 1990)

This triangle has the largest lattice width of any hollow convex body in $\mathbb{R}^2$; that is, $w_c(2) = 1 + \frac{2}{\sqrt{3}}$.

$w_c(3)$: A wide tetrahedron

In the (affine) lattice $\{(a, b, c) : a, b, c \in 1 + 2\mathbb{Z}, a + b + c \in 1 + 4\mathbb{Z}\}$,

$$T = \text{conv}\{(-1, 1, 1), (-1, -1, -1), (1, -1, 1), (1, 1, -1)\}$$

is a unimodular tetrahedron.

$w_c(3)$: A wide tetrahedron

In the (affine) lattice $\{(a, b, c) : a, b, c \in 1 + 2\mathbb{Z}, a + b + c \in 1 + 4\mathbb{Z}\}$,

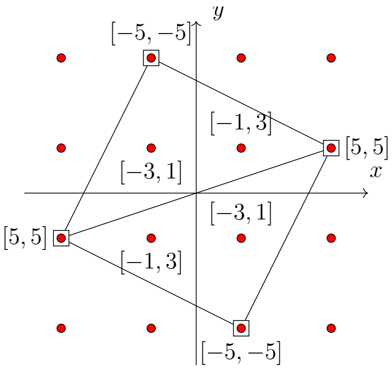

Figure: Δ_0

$$T = \operatorname{conv}\{(-1, 1, 1), (-1, -1, -1), (1, -1, 1), (1, 1, -1)\}$$

is a unimodular tetrahedron.

Δ_0 is the lattice tetrahedron circumscribed to T with vertices

$$A = (3, 1, 5),$$
$$B = (-1, 3, -5),$$
$$C = (-3, -1, 5),$$
$$D = (1, -3, -5).$$

It has width 3.

$w_c(3)$: A wide tetrahedron

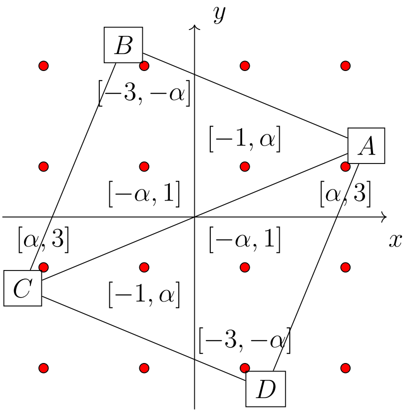

Figure: Δ has width $2 + \sqrt{2}$

We can modify Δ_0 to a tetrahedron Δ of width $2 + \sqrt{2}$.
Thus,

Corollary (C.-Santos, 2018+)

$w_c(3) \geq 2 + \sqrt{2}$.

$w_c(3)$: A wide tetrahedron

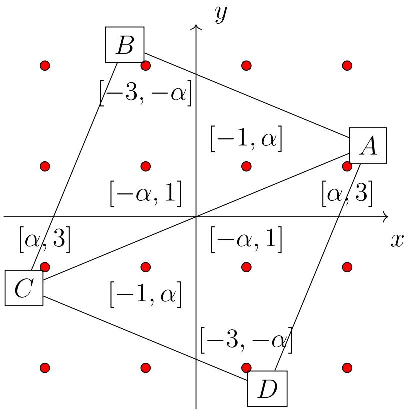

Figure: Δ has width $2+\sqrt{2}$

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Conjecture

This is the hollow 3-body of maximum width. That is, $w_c(3) = 2+\sqrt{2}$.

Interlude: Direct sum of convex bodies

Definition

Let $C_i \subset \mathbb{R}^{d_i}$ be convex bodies containing the origin. Their **direct sum** is the following convex body in $\mathbb{R}^{d_1+\cdots+d_m}$:

$$C_1 \oplus \cdots \oplus C_m =$$

$$= \operatorname{conv}\left(\bigcup_{i=1}^{m}(0 \times \cdots \times 0 \times C_i \times 0 \times \cdots \times 0)\right)$$

$$= \left\{(\lambda_1 x_1, \ldots, \lambda_m x_m) : x_i \in C_i, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1\right\}$$

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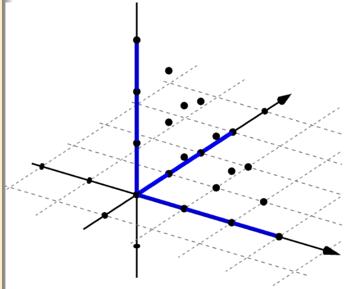

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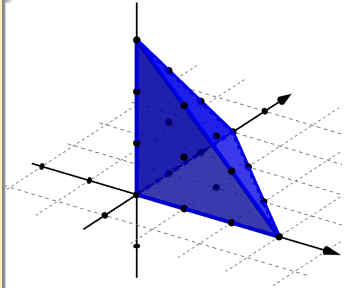

Interlude: Properties of the direct sum

It's easy to compute the width of direct sums:

$$\mathrm{width}(C_1 \oplus \cdots \oplus C_m) = \min_i \mathrm{width}(C_i),$$

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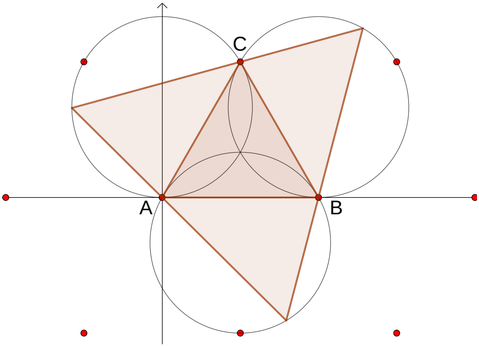
Regarding hollowness:

Lemma ((special case of) Averkov-Basu 2015)

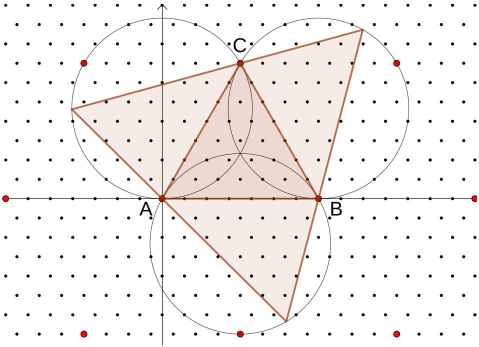
If C is hollow, then $\bigoplus_{i=1}^m mC$ is hollow of width $m \cdot \mathrm{width}(C)$.

$w_p(14)$: A lattice polytope of large width



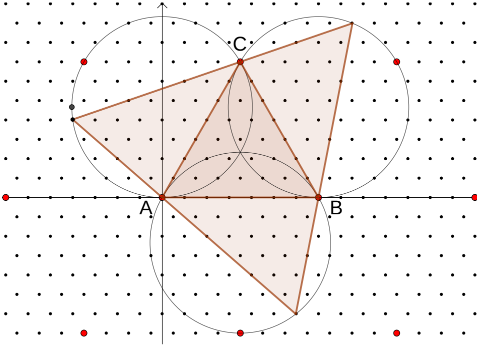
Hurkens' triangle, circumscribed to the unimodular triangle ABC

$w_p(14)$: A lattice polytope of large width



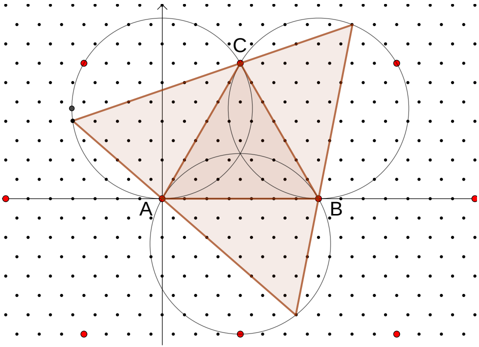
We refine the lattice, in black we have $\Lambda' = \frac{1}{7}\Lambda$

$w_p(14)$: A lattice polytope of large width



Hurkens' triangle has a nice *rational* approximation T of width $15/7 = 2.1429$.

$w_p(14)$: A lattice polytope of large width



Hurkens' triangle has a nice *rational* approximation T of width $15/7 = 2.1429$.

From the direct sum construction, with summands equal to $7T$, follows

Corollary (C.-Santos 2018+)

There is a 14-dimensional hollow lattice polytope of width 15. It has 21 vertices and $2^7 + 7$ facets.

$w_s(404)$: A lattice simplex of large width

Regarding lattice simplices, we can prove the following:

Lemma

There is a rational hollow 4-simplex of width $4\left(1+\frac{1}{101}\right)$.

This is obtained by "pushing out" a facet of a known empty lattice 4-simplex of width 4.

$w_s(404)$: A lattice simplex of large width

Regarding lattice simplices, we can prove the following:

Lemma

There is a rational hollow 4-simplex of width $4\left(1+\frac{1}{101}\right)$.

This is obtained by "pushing out" a facet of a known empty lattice 4-simplex of width 4. From the direct sum construction, we obtain

Corollary (C.-Santos 2018+)

There is a hollow 404-simplex of width 408.

Asymptotics of w_c, w_p and w_s

We also apply the direct sum construction to obtain asymptotics of our constants. In particular,

Theorem ((Codenotti-S. 2018+))

$$\lim_{d\to\infty} \frac{w_p(d)}{d} = \lim_{d\to\infty} \frac{w_c(d)}{d} = \sup_{d\to\infty} \frac{w_c(d)}{d} \geq \frac{2+\sqrt{2}}{3} = 1.138\dots,$$

$$\lim_{d\to\infty} \frac{w_s(d)}{d} \geq \frac{102}{101} = 1.0099\dots$$

where on the right we have used $w_c(3)/3 \geq \frac{2+\sqrt{2}}{3}$, based on the tetrahedron of width $2+\sqrt{2}$, and $w_s(404)/404 \geq \frac{102}{101}$ from the 404-dimensional simplex.

Thank you for your attention!

Giulia Codenotti, Francisco Santos. Hollow polytopes of large width.
Preprint, 17 pages, December 2018.
http://arxiv.org/abs/1812.00916