GAC Summer School

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The minimal cellular resolution of the edge ideals of forests

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joint work with Margherita Barile (Università degli Studi di Bari)

1.1. Minimal free resolutions

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where $F_i = \bigoplus_j S(-j)^{\beta_{ij}(S/I)} \neq 0$ and $\mathrm{Im}(\varphi_i) \subset (x_1, \dots, x_n)F_{i-1}$.

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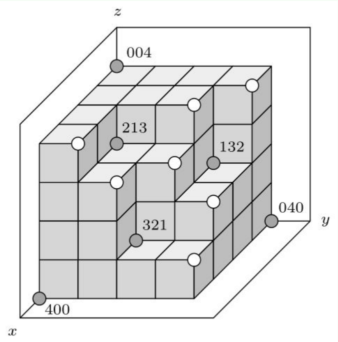
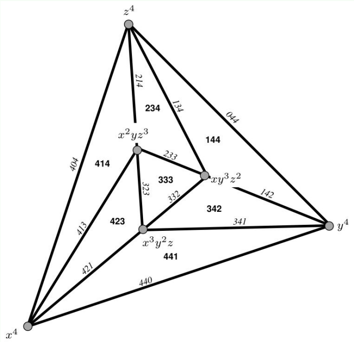
- $pd(S/I) = h \to$ projective dimension of S/I

Problem

- *Find classes of ideals whose minimal resolutions can be easily described.*
- *Describe non-minimal resolutions for large classes of monomial ideals.*

Example Let $I = (x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3) \subset S = k[x, y, z]$.
We can draw a 3-dimensional staircase diagram to represent I.
Then we draw a graph in which we connect two minimal generators of I when they "look adjacent". We label each edge and each triangular face according to the exponent vector of the lcm of its vertices.





The minimal free resolution of I can be read off this figure:

$$0 \longrightarrow S^7 \longrightarrow S^{12} \longrightarrow S^6 \longrightarrow I \longrightarrow 0,$$

where the exponents correspond to **6** vertices, **12** edges and **7** triangles.

1.2. Simplicial resolutions

Simplicial resolutions were introduced by [Bayer, Peeva, Sturmfels, 1998].

Let Δ be a simplicial complex whose vertices are labeled by the minimal generators of a monomial ideal $I \subset S$. Each face F of Δ is labeled by the least common multiple of its vertices, denoted by $\mathbf{m}_F = \mathbf{x}^{\mathbf{a}_F}$.

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Let $\mathcal{F}_\Delta$ be the reduced chain complex of Δ over S defined by

$$\mathcal{F}_\Delta = \bigoplus_{F \in \Delta} S(-\mathbf{a}_F), \quad \varphi(F) = \sum_{G \text{ facet of } F} \operatorname{sign}(G, F)\mathbf{x}^{\mathbf{a}_F - \mathbf{a}_G} G,$$

where F, G are thought of both as faces of Δ and as basis vectors in degrees $\mathbf{a}_F$ and $\mathbf{a}_G$. Moreover, $\operatorname{sign}(G, F)$ equals $+1$ if F's orientation induces G's orientation and -1 otherwise.

Proposition (BPS, 1998) *The complex $\mathcal{F}_{\Delta}$ is exact and defines a free resolution of I if and only if for every monomial $\mathbf{m}$, the complex*

$$\Delta_{\preceq \mathbf{m}} = \{F \in \Delta : \mathbf{m}_F \text{ divides } \mathbf{m}\}$$

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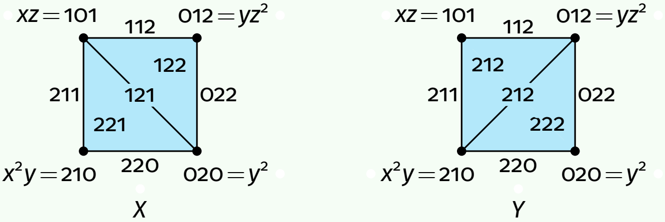
Example Let $I = (x^2y, xz, yz^2, y^2) \subset S = k[x,y,z]$ and consider the simplicial complexes X and Y on the generators of I:



X supports the free resolution $\mathcal{F}_X : 0 \to S^2 \to S^5 \to S^4 \to I \to 0.$

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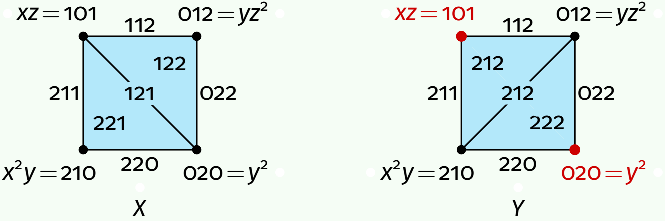

X supports the free resolution $\mathcal{F}_X : 0 \to S^2 \to S^5 \to S^4 \to I \to 0$. Y does not support a free resolution of I: $Y_{\preceq xy^2z}$ consists of the two vertices xz and y^2, hence it is not acyclic.

1.3. Minimality

Proposition (BPS, 1998) *Let $\mathcal{F}_\Delta$ be a free resolution of the monomial ideal I supported by the labeled simplicial complex Δ. Then $\mathcal{F}_\Delta$ is a minimal free resolution if and only if any two comparable faces $G \subset F$ of Δ have distinct degrees, i.e. $\mathbf{m}_G \neq \mathbf{m}_F$.*

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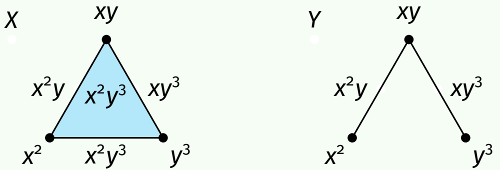

Both X and Y support a free resolution of I:

$$\mathcal{F}_X : 0 \to S \to S^3 \to S^3 \to I \to 0 \qquad \mathcal{F}_Y : 0 \to S^2 \to S^3 \to I \to 0$$

$\mathcal{F}_Y$ is minimal; $\mathcal{F}_X$ is not minimal: the triangle and one of its edges have the same label x^2y^3.

1.4. The Taylor resolution (1966)

If I is a monomial ideal with r minimal generators, let Δ be the full $(r-1)$-dimensional simplex whose r vertices are labeled by the generators $\mathbf{m}_1, \dots, \mathbf{m}_s$.

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In particular, $\Delta_{\preceq \mathbf{m}}$ is contractible, hence acyclic. Thus $\mathcal{F}_\Delta$ is a simplicial resolution of I called Taylor resolution.

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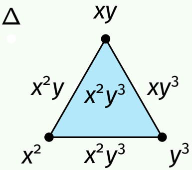
Example Let $I = (x^2, xy, y^3) \subset S = k[x, y]$. The Taylor resolution of I is



$$\mathcal{F}_\Delta : 0 \to S \to S^3 \to S^3 \to I \to 0$$

The Taylor resolution is often not minimal.

1.5. Make a cellular resolution minimal

Using Discrete Morse Theory, one can hope to prune a cellular resolution to get a minimal one. Algorithms by

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Minimal cellular resolutions have been found for

- generic and shellable monomial modules [Batzies, Welker, 2002],
- the powers of the edge ideals of paths [Engström, Norén, 2012],
- the Eliahou-Kervaire resolution for stable ideals [Mermin, 2010],
- the matroid ideal of a finite projective space [Novik, 2002].

2.1. Edge ideals of trees

Definition Let G be a graph on the vertex set $[n]$ and $S = k[x_1, \dots, x_n]$. The edge ideal of G is the squarefree monomial ideal

$$I(G) = (x_i x_j : \{i, j\} \in E(G)) \subset S.$$

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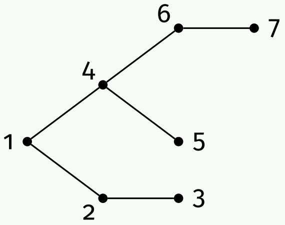
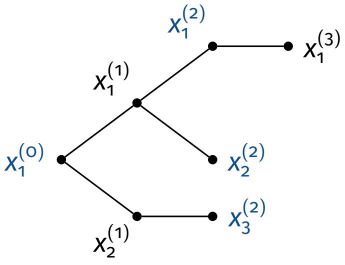
Here we consider **trees** that are connected graphs without cycles.

Example Let T be the following tree:



$$I(T) = \begin{pmatrix} x_1x_2, x_1x_4, x_2x_3, \\ x_4x_5, x_4x_6, x_6x_7 \end{pmatrix} \subset k[x_1, \dots, x_7]$$

Fix a vertex $x_1^{(0)}$ of T and call $x_1^{(i)}, \dots, x_{s_i}^{(i)}$ the vertices lying at distance i from $x_1^{(0)}$. Consider the lexicographic order induced by
$x_1^{(0)} > x_1^{(1)} > \cdots > x_{s_1}^{(1)} > \cdots > x_1^{(d)} > \cdots > x_{s_d}^{(d)}$.



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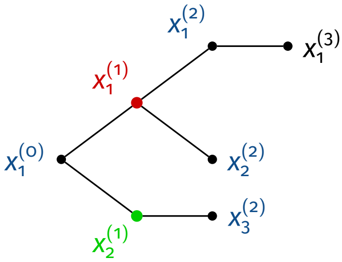

1. Select a descending sequence of variables $x_{p_1}^{(i_1)}, \dots, x_{p_t}^{(i_t)}$ corresponding to pairwise non-adjacent vertices of T.

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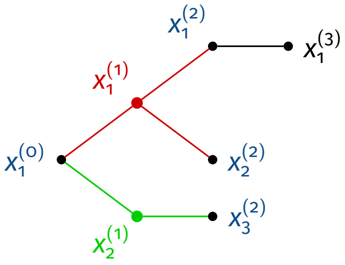

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2. Pick all edge monomials divisible by one of the variables $x_{p_1}^{(i_1)}, \dots, x_{p_t}^{(i_t)}$.

$$x_1^{(0)}x_1^{(1)}, \quad x_1^{(0)}x_2^{(1)}, \quad x_1^{(1)}x_1^{(2)}, \quad x_1^{(1)}x_2^{(2)}, \quad x_2^{(1)}x_3^{(2)}$$

3. Remove all monomials μ that are divisible by $x_{p_h}^{(i_h)}$ and are not co-prime with respect to an element ν of the symbol divisible by $x_{p_k}^{(i_k)}$ for some $k > h$.

$$\cancel{x_1^{(0)} x_1^{(1)}}, \quad x_1^{(0)} x_2^{(1)}, \quad x_1^{(1)} x_1^{(2)}, \quad x_1^{(1)} x_2^{(2)}, \quad x_2^{(1)} x_3^{(2)}$$

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4 Consider all subsymbols of the symbols obtained so far.

$$[x_1^{(0)}x_2^{(1)}, x_1^{(1)}x_1^{(2)}, x_1^{(1)}x_2^{(2)}], \quad [x_1^{(0)}x_2^{(1)}, x_1^{(1)}x_1^{(2)}, x_1^{(1)}x_3^{(2)}],$$

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Definition Let xy and zw be elements of a symbol $\mathbf{u}$, where $xy > zw$. If $xz \in I$, we say that xz is the **bridge** between xy and zw.
In this case, we will say that xy and zw form a **gap** in $\mathbf{u}$ if $xz \notin \mathbf{u}$, no other monomial of $\mathbf{u}$ other than zw is divisible by w, and no monomial smaller than zw is divisible by y.

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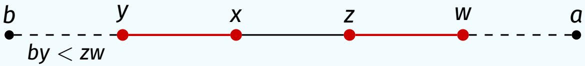

The last step of the procedure is

5. Discard all symbols that contain a gap.

~~$[x_1^{(0)}x_2^{(1)}, x_1^{(1)}x_1^{(2)}, x_1^{(1)}x_2^{(2)}]$~~, $[x_1^{(0)}x_2^{(1)}, x_1^{(1)}x_1^{(2)}, x_1^{(1)}x_3^{(2)}]$

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The symbols obtained after 1-5 are called *F*-**admissible** for T.

Counting all F-admissible symbols of length r and degree d we can compute the graded Betti numbers $\beta_{r,d}(S/I(T)) = \beta_{r-1,d}(I(T))$:

$$0 \to S(-6)^2 \to S(-4) \oplus S(-5)^6 \to S(-3)^6 \oplus S(-4)^4 \to S(-2)^6 \to I(T) \to 0.$$

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For every r, let F_r be the free S-module generated by the F-admissible symbols of length r.

We show that F_r is the r-th module of a minimal graded free resolution of $S/I(T)$.

2.4. Minimal cellular resolution

Proposition (Batzies, Welker, 2002) *There exists a CW-complex that is homotopy equivalent to the Taylor resolution and whose r-cells are in 1-to-1 correspondence with the F-admissible symbols of length r.*

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[Batzies, Welker, 2002] describe explicitly the differentials $\partial_r : F_r \to F_{r-1}$ in terms of the F-admissible symbols.

Theorem *The cellular resolution (F_r, ∂_r) is a minimal graded free resolution of $S/I(T)$.*

We also found a new elementary proof of Jacques' recursive formulas for the graded Betti numbers and for the projective dimension.

Any tree T contains a vertex v such that among its neighbours $v_1, \dots, v_m$ at most one (say v_m) has degree greater than 1. Then call $T' = T \setminus \{v_1\}$ and $T'' = T \setminus \{v, v_1, \dots, v_m\}$.

Theorem (Jacques, 2004) *For all indices r, d we have*

$$\beta_{r,d}(S/I(T)) = \beta_{r,d}(S/I(T')) + \sum_{j=0}^{m-1} \binom{m-1}{j} \beta_{r-(j+1),d-(j+2)}(S/I(T'')).$$

Moreover,

$$\mathrm{pd}(S/I(T)) = \max\{\mathrm{pd}(S/I(T')), \mathrm{pd}(S/I(T'')) + m\}.$$

Thank you for listening!