Envy-free division of a cake: the poisoned case, and other variations

Frédéric Meunier

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Joint work with Shira Zerbib

Genesis, Chapter 13

From the Negev Abram went from place to place until he came to Bethel, to the place between Bethel and Ai where his tent had been earlier and where he had first built an altar. There Abram called on the name of the Lord.

Now Lot, who was moving about with Abram, also had flocks and herds and tents. But the land could not support them while they stayed together, for their possessions were so great that they were not able to stay together. And quarreling arose between Abram's herders and Lot's. The Canaanites and Perizzites were also living in the land at that time.

So Abram said to Lot, "Let's not have any quarreling between you and me, or between your herders and mine, for we are close relatives. Is not the whole land before you? Let's part company. If you go to the left, I'll go to the right; if you go to the right, I'll go to the left."

Dividing a cake

Theorem (Abraham, 1850 BC)

To divide a cake between two people in an envy-free manner, let one person cut the cake and let the other choose.

Envy-free cake cutting: the traditional setting

Envy-free cake sharing

A cake has to be shared between people.

It will be divided into as many pieces as there are people.

Each person will be assigned a piece.

Envy-free sharing of a cake: each person prefers his piece.

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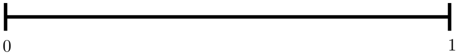
Envy-free sharing of a cake: each person is at least as happy with his piece than with any other piece.

Model

- n players: $i = 1, \ldots, n$

- Cake: $[0, 1] =$ 

- Division of the cake: partition $\mathcal{I}$ of $[0, 1]$ into pairwise disjoint nonempty intervals, called pieces. (⚠ Assumption: boundary points do not matter)

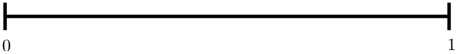
- Player i has a preference function:

$$p_i \colon \{\text{divisions}\} \to 2^{\{\text{pieces}\}}$$

Meaning: *Given a division $\mathcal{I}$, player i is happy with the pieces $I \in \mathcal{I}$ such that $I \in p_i(\mathcal{I})$.*

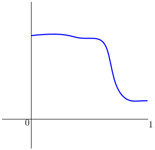
- Envy-free sharing: division $\mathcal{I}$ and assignment $\pi \colon \{\text{players}\} \longrightarrow \{\text{pieces}\}$ such that
  - ⋆ π is bijective.
  - ⋆ $\pi(i) \in p_i(\mathcal{I})$ for every player i.

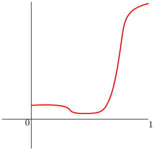
Example with 2 players

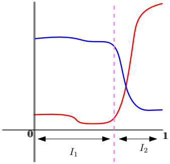
- 2 players: Alice and Bob
- Cake: $[0,1] =$ 

- Example

$$\mu_X(I) = \int_I f_X(u)\mathrm{d}u \qquad p_X(\mathcal{I}) = \arg\max\{\mu_X(I) \colon I \in \mathcal{I}\}.$$

with $f_A =$ 

$f_B =$ 

- Sharing: 

$\pi(A) = I_1$, $\pi(B) = I_2$

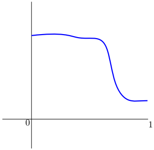
Example with 3 players

- 3 players: Alice, Bob, and Charlie

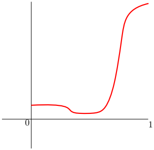
- Example

$$\mu_X(I) = \int_I f_X(u)\mathrm{d}u \qquad p_X(\mathcal{I}) = \arg\max\{\mu_X(I)\colon\ I \in \mathcal{I}\},$$

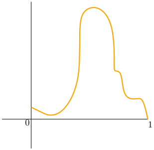
$f_A =$



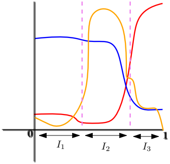
$f_B =$



$f_C =$



- Sharing:



$\pi(A) = I_1,\ \pi(B) = I_3,\ \pi(C) = I_2.$

Existence of envy-free divisions

- Preference function p_i is closed if

$$\lim_{k\to\infty} \mathcal{I}^k = \mathcal{I} \text{ and } I^k \in p_i(\mathcal{I}^k)\ \forall k \quad \Longrightarrow \quad I^\infty \in p_i(\mathcal{I})$$

- Preference function p_i is hungry if

$$I \in p_i(\mathcal{I}) \quad \Longrightarrow \quad \lambda(I) \neq 0$$

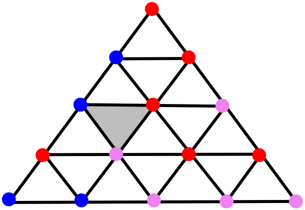
Theorem (Stromquist, Woodall, 1980)

No matter how many players there are, when all preference functions are closed and hungry, there is always an envy-free sharing.

Constructive proof

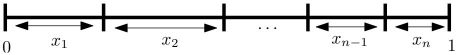
Su (1998) proposed an elegant proof based on Sperner's lemma.

⇒ algorithmic proof (path-following, pivot) for finding an approximate envy-free sharing.



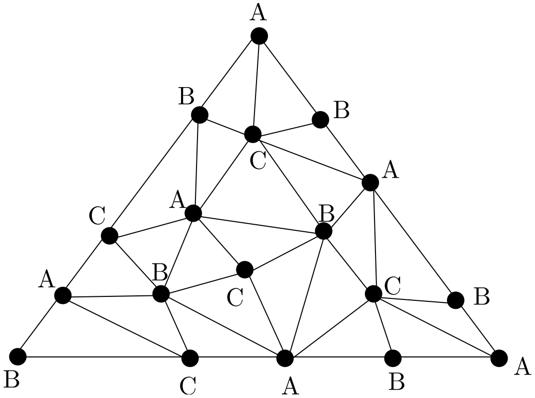
Sperner's lemma

Space of division:
$\triangle^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}_+^n : \sum_{i=1}^n x_i = 1\}$



Focus on Su's proof

Three players: A,B,C.



Ask each player which piece she prefers.

Focus on Su's proof

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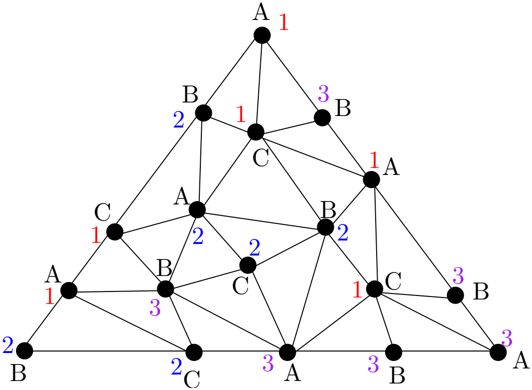

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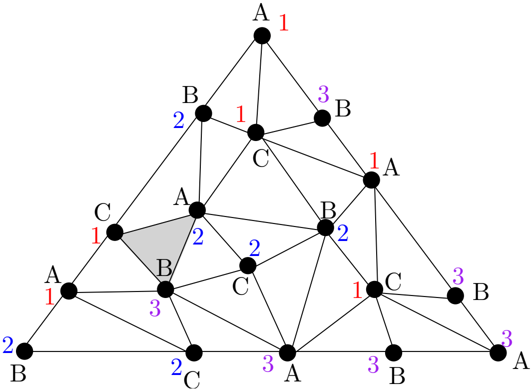
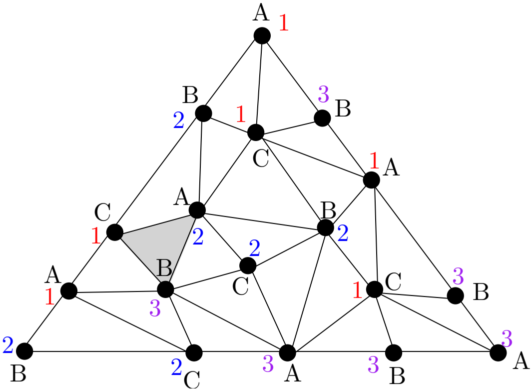


p_i's are hungry $\Longrightarrow$ Sperner's lemma: There is an approximate envy-free sharing.

Focus on Su's proof

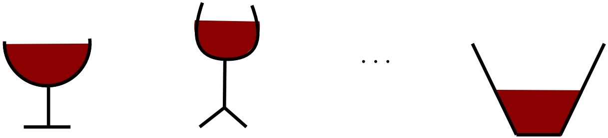
Three players: A,B,C.



Compactness and p_i's are closed: There is an envy-free sharing.

Wine

Other things than lands and cakes can be shared.



Complexity remarks

Three theorems by Deng, Qi, and Saberi (2012).

Theorem

Finding an ε-approximation of an envy-free sharing of the cake is PPAD-complete.

Theorem

The query complexity of finding an ε-approximation is $\theta((1/\varepsilon)^{n-1})$.

Theorem

If the preference functions are Lipschitz and monotone, then there is an FPTAS for the case with $n = 3$ players.

The case $n > 3$ remains open.

The poisoned case

Preference functions can model more

Player i's preference function: $p_i \colon \{\text{divisions}\} \to 2^{\{\text{pieces}\}}$.
Given a division $\mathcal{I}$, player i is happy with the pieces I such that $I \in p_i(\mathcal{I})$.

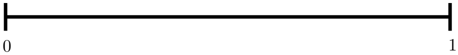
It is very flexible: no monotonicity assumption, can make the preferences depend on all pieces, etc.

Other example:

$$\mu_X(I) = \int_I f_X(u)\mathrm{d}u \qquad p_X(\mathcal{I}) = \begin{cases} \arg\min\{\mu_X(I) \colon I \in \mathcal{I}\} & \text{if } |\mathcal{I}| = n \\ \varnothing & \text{otherwise.} \end{cases}$$

Can be used to model burnt or poisoned cake.

More general model

- n players: $i = 1, \ldots, n$
- Cake: $[0, 1] =$ 

- Division of the cake: partition $\mathcal{I}$ of $[0, 1]$ into pairwise disjoint nonempty intervals, called pieces. (Assumption: boundary points do not matter)
- Player i has a preference function: $p_i : \{\text{divisions}\} \to 2^{\{\text{pieces}\} \cup \{\varnothing\}}$

Meaning: *Given a division $\mathcal{I}$, player i is happy*
  - *either with the pieces $I \in \mathcal{I}$ such that $I \in p_i(\mathcal{I})$,*
  - *or with $\varnothing$.*
- Envy-free sharing: division $\mathcal{I}$ and assignment

$\pi : \{\text{players}\} \longrightarrow \{\text{pieces}\} \cup \{\varnothing\}$ such that
  - ⋆ For every piece I, there exists a unique player i s.t. $\pi(i) = I$ ("bijectivity")
  - ⋆ $\pi(i) \in p_i(\mathcal{I})$ for every player i.

A more general theorem

♣ Preference function p_i is closed if

$$\lim_{k\to\infty} \mathcal{I}^k = \mathcal{I} \text{ and } I^k \in p_i(\mathcal{I}^k)\ \forall k \quad \Longrightarrow \quad I^\infty \in p_i(\mathcal{I})$$

♣ Full-division assumption: when the cake is divided into n pieces, no player is happy to get nothing.

Theorem (M., Zerbib 2019)

Consider an instance with n players, with closed preference functions and the full-division assumption. If n is a prime number or is equal to 4, then there exists an envy-free division of the cake.

Conjectured by Segal-Halevi (2018) to be true for all n.

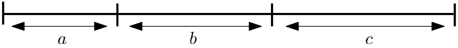
Cases $n = 1, 2, 3$

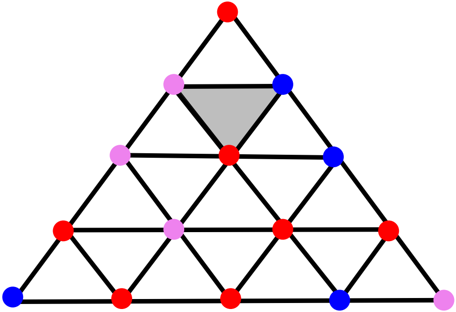
- Case $n = 1$ is obvious
- Case $n = 2$: A and B
  - there is a division into two nonempty intervals between which A is indifferent:
    - start in the middle
    - move in each direction: the limit division is the same in both cases
    - use continuity to conclude
  - A cuts, B chooses
- Case $n = 3$, proved by Segal-Halevi (2018).

Proof of the theorem

If 3 is a preferred piece in a division
$(x_1, x_2, x_3, x_4, x_5) = (0, a, b, 0, c)$,

then 4 is a preferred piece in the division
$(x_1, x_2, x_3, x_4, x_5) = (a, 0, 0, b, c)$.

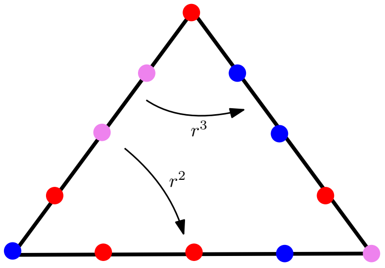




$\Longrightarrow$ "Sperner's lemma" with a symmetry on the boundary

proved by Segal-Halevi (2018) for $n = 3$.

Sperner's lemma with a symmetry

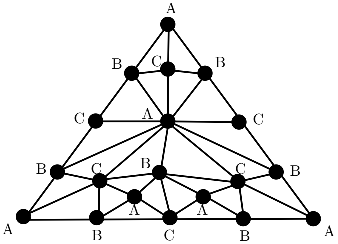


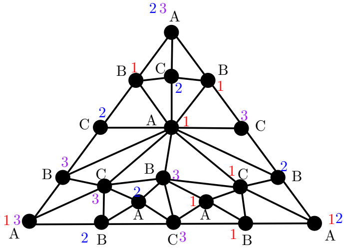
Triangulation T and labeling Λ are nice if $\Lambda\big(r^j(v)\big) = \rho^j\big(\Lambda(v)\big)$ or every $j \in [n]$ and every vertex v of the $\widehat{1}$-facet.

Theorem

Let T *be a nice triangulation of* Δ^{n-1} *and let* Λ *be a nice labeling of its vertices with nonempty proper subsets of* $[n]$*. If* n *is a prime number, then there is an* $(n-1)$*-dimensional simplex* $\tau \in$ T *such that it is possible to pick a distinct label in each* $\Lambda(u)$ *when* u *runs over the vertices of* τ.

Use of “Sperner’s lemma with symmetry”



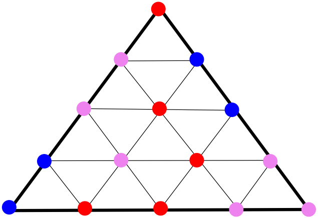


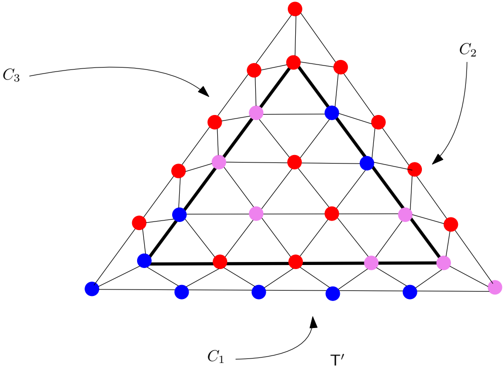
$$\Lambda(\boldsymbol{x}) = \begin{cases} \{\text{pieces preferred by } \boldsymbol{x}\} & \text{if player prefers to have a piece} \\ \{i \in [n] \colon x_i = 0\} & \text{otherwise.} \end{cases}$$

Full-division assumption: every vertex gets a label

Sperner’s lemma with symmetry + (compactness and p_i’s are closed): there exists an envy-free sharing

Proof of “Sperner’s lemma with symmetry”





$$\mathrm{T}' = \mathrm{T} \cup \left(\bigcup_{j=1}^{n} C_j \right)$$

Lemma (M., Zerbib 2019)

Let λ be a labeling of a triangulation K *of Δ^{n-1}. If $\lambda(v)$ belongs to the supporting face of v for all vertices v, then*
$$\left| \sum_{[v_1,\ldots,v_n]\in \mathrm{K}} \det(\lambda(v_1), \ldots, \lambda(v_n)) \right| = 1.$$

Proof of "Sperner's lemma with symmetry"

$$t' = t + \sum_{j=1}^{n} c_j \quad \text{with} \quad \begin{cases} t' = \sum_{[v_1,\ldots,v_n] \in \mathrm{T}'} \det(\lambda(v_1),\ldots,\lambda(v_n)) \\ t = \sum_{[v_1,\ldots,v_n] \in \mathrm{T}} \det(\lambda(v_1),\ldots,\lambda(v_n)) \\ c_j = \sum_{[v_1,\ldots,v_n] \in C_j} \det(\lambda(v_1),\ldots,\lambda(v_n)) \end{cases}$$

Symmetry $\Longrightarrow \sum_{j=1}^{n} c_j = nc_1$

$\lambda(v)$ is of the form $\frac{1}{|\Lambda(v)|}\overrightarrow{\chi}^{\Lambda(v)}$ where $\chi_i^S = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$

A generalization of Sperner's lemma $\Longrightarrow |t'| = 1$

n prime $\Longrightarrow t \neq 0$ $\square$

Remarks

♣ Case n nonprime and $\neq 4$ is still open.

♣ If goods have to be shared, true for all n: classical cake-division theorem. If chores have to be shared, true for all n: rental-harmony theorem. Mixing makes the problem difficult.

♣ "Sperner's lemma with symmetry" looks like a combinatorial fixed-point theorem for higher dimensional generalizations of the dunce hat space. Other applications? Nice interpretation?

♣ Research activity on discrete versions of cake-divisions.

Thank you.