

Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics

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H. S. M. Coxeter, *Frieze patterns*, Acta Arith. **18** (1971), 297–310:

“The idea of frieze pattern is most quickly conveyed by means of an example”

Coxeter frieze pattern - Example

...	1	1	1	1	1	1	1	1	1	1
	2	1	3	2	2	1	4	2	1	...
...	1	2	5	3	1	3	7	1	2	
	3	1	3	7	1	2	5	3	1	...
...	2	1	4	2	1	3	2	2	1	
	1	1	1	1	1	1	1	1	1	...

Coxeter frieze pattern - Example

?	1	1	1	1	1	1	1	1	1	1
	2	1	3	2	2	1	4	2	1	...
?	1	2	5	3	1	3	7	1	2	
	3	1	3	7	1	2	5	3	1	...
?	2	1	4	2	1	3	2	2	1	
	1	1	1	1	1	1	1	1	1	...

Coxeter frieze pattern - Example

1	1	1	1	1	1	1	1	1	1	1
	2	1	3	2	2	1	4	2	1	...
?	1	2	5	3	1	3	7	1	2	
	3	1	3	7	1	2	5	3	1	...
?	2	1	4	2	1	3	2	2	1	
	1	1	1	1	1	1	1	1	1	...

Coxeter frieze pattern - Example

1	1	1	1	1	1	1	1	1	1	1
	2	1	3	2	2	1	4	2	1	...
7	1	2	5	3	1	3	7	1	2	
	3	1	3	7	1	2	5	3	1	...
?	2	1	4	2	1	3	2	2	1	
	1	1	1	1	1	1	1	1	1	...

Coxeter frieze pattern - Example

1	1	1	1	1	1	1	1	1	1	1
	2	1	3	2	2	1	4	2	1	...
7	1	2	5	3	1	3	7	1	2	
	3	1	3	7	1	2	5	3	1	...
2	2	1	4	2	1	3	2	2	1	
	1	1	1	1	1	1	1	1	1	...

Coxeter frieze pattern - Periodicity

...	1	1	1	1	1	1	1	1	1	1
	2	1	3	2	2	1	4	2	1	...
...	1	2	5	3	1	3	7	1	2	
	3	1	3	7	1	2	5	3	1	...
...	2	1	4	2	1	3	2	2	1	
	1	1	1	1	1	1	1	1	1	1

Coxeter frieze pattern - Glide symmetry

	1	1	1	1	1	1	1	1	1	1	
	2	1	3	2	2	1	4	2	1	...	
...	1	2	5	3	1	3	7	1	2		
	3	1	3	7	1	2	5	3	1	...	
...	2	1	4	2	1	3	2	2	1		
	1	1	1	1	1	1	1	1	1	1	

SL₂-friezes

	1	1	1	1	1	1	1	1	1	1	...
...		7		2	2	2	5	3	2	2	
	11		10		2	2	8	5	5	1	...
...		15		7	1	7	5	6	2	2	
	2		10		1	3	3	4	2	2	...
...		1		1	1	1	1	1	1	1	

	1	1	1	1	1	1	1	1	1	1	...
...		2	4		4	3	1	4	10	1	2
	1		3		10	6	1	3	10	6	1
...		1		4		10	1	2	4	4	3
	1		1		1	1	1	1	1	1	1
...		1		1		1	1	1	1	1	1

SL_?-friezes

SL₃-frieze

	1	1	1	1	1	1	1	1	1	...	
...		7		2	2	2	5	3	2	2	
	11		10		2	2	8	5	5	1	...
...		15		7		1	7	5	6	2	2
	2		10		1	3	3	4	2	2	...
...		1		1		1	1	1	1	1	1

	1	1	1	1	1	1	1	1	1	1	...
...		2	4		4	3	1	4	10	1	2
	1	3	10		6	1	3	10	6	1	...
...		1	4		10	1	2	4	4	3	1
	1	1	1		1	1	1	1	1	1	...

$SL_?$ -friezes

SL_3 -frieze

	1	1	1	1	1	1	1	1	1	...
...	7	2	2	2	5	3	2	2		
	11	10	2	2	8	5	5	1	...	
...	15	7	1	7	5	6	2	2		
	2	10	1	3	3	4	2	2	...	
...	1	1	1	1	1	1	1	1		

SL_4 -frieze

	1	1	1	1	1	1	1	1	1	1	...
...	2	4	4	3	1	4	10	1	2		
	1	3	10	6	1	3	10	6	1	...	
...	1	4	10	1	2	4	4	3	1		
	1	1	1	1	1	1	1	1	1	...	

1		1		1		1		1		1		1		1	
	7		2		2		2		5		3		2		2
11		10		2		2		8		5		5		1	
	15		7		1		7		5		6		2		2
2		10		1		3		3		4		2		2	
	1		1		1		1		1		1		1		1

1		1		1		1		1		1		1		1
	7	4	2		2	2		5		3		2		2
11		10		2		2		8		5		5		1
	15		7		1		7		5		6		2	2
2		10		1		3		3		4		2		2
	1		1		1		1		1		1		1	1

1		1		1		1		1		1		1		1
	7	4	2	2	2		2		5		3		2	2
11		10		2		2		8		5		5		1
	15		7		1		7		5		6		2	2
2		10		1		3		3		4		2		2
	1		1		1		1		1		1		1	1

1		1		1		1		1		1		1		1
	7	4	2	2	2		2		5		3		2	2
11		10	6	2		2		8		5		5		1
	15		7		1		7		5		6		2	2
2		10		1		3		3		4		2		2
	1		1		1		1		1		1		1	1

1		1	1	1		1		1		1		1	
	7	4	2	2	2		2		5		3	2	2
11		10	6	2		2		8		5		5	1
	15		7		1		7		5		6	2	2
2		10		1		3		3		4		2	2
	1		1		1		1		1		1	1	1

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
3	7	4	2	2	2	2	2	5	10	3	1	2	3	2	
11	5	10	6	2	2	2	2	8	15	5	7	5	1	1	7
8	15	5	7	5	1	1	7	11	5	10	6	2	2	2	2
2	5	10	3	1	2	3	2	3	7	4	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
3	7	4	2	2	2	2	2	2	5	10	3	1	2	3	2
11	5	10	6	2	2	2	2	8	15	5	7	5	1	1	7
8	15	5	7	5	1	1	7	11	5	10	6	2	2	2	2
2	5	10	3	1	2	3	2	3	7	4	2	2	2	2	2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

→ 2-frieze

	1	1	1	1	1	1	1	1	1	1	1	1	...
...	2	4	4	3	1	4	10	1	2	4	3	1	...
	1	3	10	6	1	3	10	6	1	1	1	...	
...	1	4	10	1	2	4	4	3	1	1	1	...	
	1	1	1	1	1	1	1	1	1	1	1	...	

	1	1	1	1	1		1		1		1		1		1		1	...
...	1	2	5	4	6	4		3		1		4	10		1		2	
	1	1	3	14	10		6		1		3		10		6		1	...
...		1	1	4		10		1		2		4		4		3		1
	1		1		1		1		1		1		1		1		1	...

Symplectic 2-frieze

...	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	...	
...	1	2	5	4	6	4	6	3	2	1	1	4	30	10	4	1	1	2	...
...	1	1	3	14	10	20	6	3	1	1	3	14	10	20	6	3	1	1	...
...	2	1	1	4	30	10	4	1	1	2	5	4	6	4	6	3	2	1	...
...	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	...

More friezes

- additive: $a + d = b + c$ or $a + d = b + c + 1$

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- tropical: $a + d = \max(b + c, 0)$ or
 $a + d = \max(b, 0) + \max(c, 0)$

More friezes

- additive: $a + d = b + c$ or $a + d = b + c + 1$
- tropical: $a + d = \max(b + c, 0)$ or
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- quantum: $ad - q^{\frac{1}{2}}bc = 1$

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- quantum: $ad - q^{\frac{1}{2}}bc = 1$
- NIM: $a \boxplus d = b \boxplus c + 1$,
- continuous: $F(x, y): F \frac{\partial^2}{\partial x \partial y} F - \frac{\partial}{\partial x} F \frac{\partial}{\partial y} F = 1$

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- SL_k -frieze or SL_k -tiling

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- continuous: $F(x, y): F \frac{\partial^2}{\partial x \partial y} F - \frac{\partial}{\partial x} F \frac{\partial}{\partial y} F = 1$
- infinite frieze (remove the boundary rows of 1's)
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- frieze over a repeated quiver
- superfrieze or $Osp(1|2)$ -frieze

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- tropical: $a + d = \max(b + c, 0)$ or
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- quantum: $ad - q^{\frac{1}{2}}bc = 1$
- NIM: $a \boxplus d = b \boxplus c + 1$,
- continuous: $F(x, y): F \frac{\partial^2}{\partial x \partial y} F - \frac{\partial}{\partial x} F \frac{\partial}{\partial y} F = 1$
- infinite frieze (remove the boundary rows of 1's)
- SL_k -frieze or SL_k -tiling
- frieze over a repeated quiver
- superfrieze or $Osp(1|2)$ -frieze
- 2-frieze
- ...

Many authors

Assem, Baur, Bergeron, Bessenrodt, Caldero, Chapoton, Cuntz, Dupont, Holm, Jørgensen, Gunawan, Keller, Marsh, Musiker, Ovsienko, Plamondon, Propp, Reutenauer, Ringel, Sherotzke, Smith, Tabachnikov, Thomas, ...

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S. Morier-Genoud, *Symplectic frieze patterns*, arXiv:1803.06001