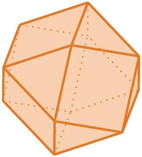
The Hopf Monoid of Orbit Polytopes

Mariel Supina

Department of Mathematics
University of California, Berkeley



June 19, 2019

Outline

1 Hopf Monoids

2 Orbit Polytopes

3 The Hopf Monoid of Orbit Polytopes

4 Character Group of Orbit Polytopes

Hopf monoids

Definition

A Hopf monoid is a set species equipped with a product and coproduct that satisfy naturality, unitality, associativity, and compatibility.

Hopf monoids

Definition

A Hopf monoid is a set species equipped with a product and coproduct that satisfy naturality, unitality, associativity, and compatibility.

Translation:

- Set species: “Family of combinatorial objects labelled by finite sets”

Hopf monoids

Definition

A Hopf monoid is a set species equipped with a product and coproduct that satisfy naturality, unitality, associativity, and compatibility.

Translation:

- Set species: "Family of combinatorial objects labelled by finite sets"
- Product: "Merge" two objects

Hopf monoids

Definition

A Hopf monoid is a set species equipped with a product and coproduct that satisfy naturality, unitality, associativity, and compatibility.

Translation:

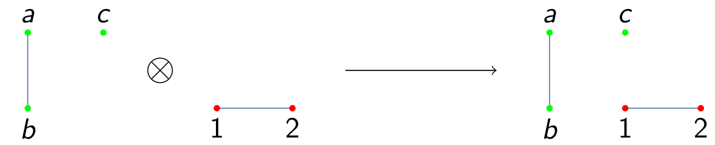
- Set species: "Family of combinatorial objects labelled by finite sets"
- Product: "Merge" two objects
- Coproduct: "Split" an object into two objects

Hopf Monoid Example: Graphs

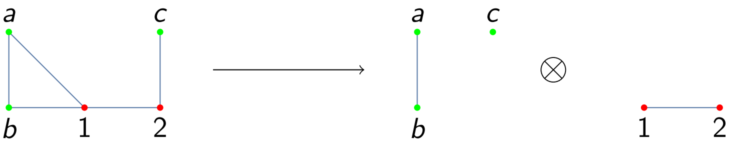
Product:

Hopf Monoid Example: Graphs

Product:



Coproduct:



The Hopf Monoid of Orbit Polytopes

Mariel Supina

Hopf Monoids

Orbit Polytopes

The Hopf Monoid of Orbit Polytopes

Character Group of Orbit Polytopes

Generalized Permutahedra

Theorem [Aguiar/Ardila 2017]

Generalized permutahedra form a Hopf monoid.

Definition

A generalized permutahedron is a polytope such that all of its edges have direction $e_i - e_j$ for some i, j.

Generalized Permutahedra

Theorem [Aguiar/Ardila 2017]

Generalized permutahedra form a Hopf monoid.

Definition

A generalized permutahedron is a polytope such that all of its edges have direction $e_i - e_j$ for some i, j.

- Matroid polytopes
- Graphic zonotopes
- Permutahedra
- Associahedra
- Orbit polytopes

Generalized Permutahedra

Theorem [Aguiar/Ardila 2017]

Generalized permutahedra form a Hopf monoid.

Definition

A generalized permutahedron is a polytope such that all of its edges have direction $e_i - e_j$ for some i, j.

- Matroid polytopes
- Graphic zonotopes
- Permutahedra
- Associahedra
- Orbit polytopes

These are all Hopf submonoids!

Hopf Monoid of Generalized Permutahedra

- Labelling comes from the labelling of the basis vectors of the ground set
- Product: Standard product on polytopes

$$P \cdot Q = \{(p, q) : p \in P, q \in Q\}$$

- Coproduct: [We will come back to this]

Hopf Monoid of Generalized Permutahedra

- Labelling comes from the labelling of the basis vectors of the ground set
- Product: Standard product on polytopes

$$P \cdot Q = \{(p, q) : p \in P, q \in Q\}$$

- Coproduct: [We will come back to this]

Theorem [Aguiar/Ardila 2017]

The character group of permutahedra is isomorphic to power series under multiplication. For associahedra, it is isomorphic to power series under composition.

Orbit Polytopes

Hopf Monoids
Orbit Polytopes
The Hopf Monoid of Orbit Polytopes
Character Group of Orbit Polytopes

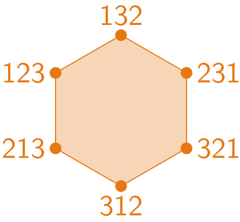
Definition

An orbit polytope has the form $\text{conv}\{\sigma(p) : \sigma \in S_n\} =: \mathcal{O}(p)$ for some point $p \in \mathbb{R}^n$.

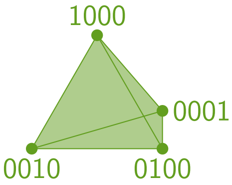
Orbit Polytopes

Definition

An orbit polytope has the form $\text{conv}\{\sigma(p) : \sigma \in S_n\} =: \mathcal{O}(p)$ for some point $p \in \mathbb{R}^n$.



$\mathcal{O}(3, 2, 1)$

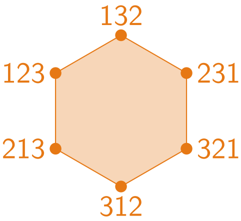


$\mathcal{O}(1, 0, 0, 0)$

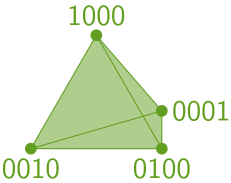
Orbit Polytopes

Definition

An orbit polytope has the form $\text{conv}\{\sigma(p) : \sigma \in S_n\} =: \mathcal{O}(p)$ for some point $p \in \mathbb{R}^n$.

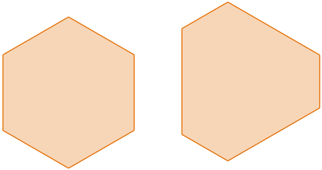


$\mathcal{O}(3, 2, 1)$

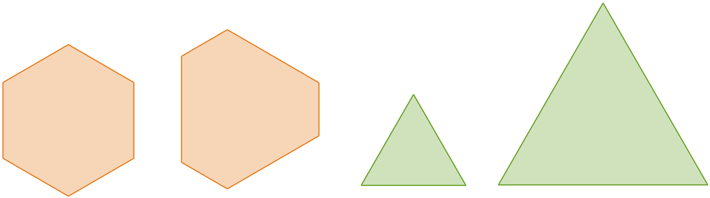


$\mathcal{O}(1, 0, 0, 0)$

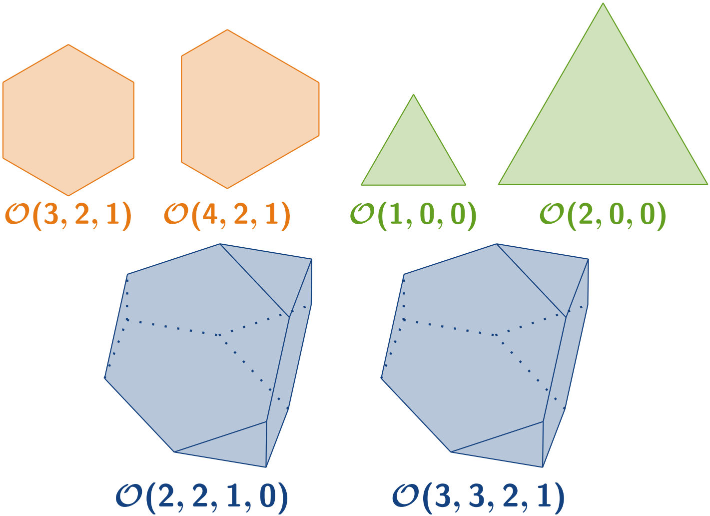
These are exactly the generalized permutahedra that are invariant under the action of the symmetric group.



$\mathcal{O}(3,2,1)$ $\mathcal{O}(4,2,1)$



$\mathcal{O}(3,2,1)$ $\mathcal{O}(4,2,1)$ $\mathcal{O}(1,0,0)$ $\mathcal{O}(2,0,0)$


$\mathcal{O}(3,2,1)$
$\mathcal{O}(4,2,1)$
$\mathcal{O}(1,0,0)$
$\mathcal{O}(2,0,0)$
$\mathcal{O}(2,2,1,0)$
$\mathcal{O}(3,3,2,1)$

Orbit Polytopes up to Normal Equivalence

Definition

The polytopes P and Q are normally equivalent if they have the same normal fan.

Orbit Polytopes up to Normal Equivalence

Definition

The polytopes P and Q are normally equivalent if they have the same normal fan.

If P and Q are orbit polytopes in $\mathbb{R}^n$, let $p \in \mathrm{vert}(P)$ and $q \in \mathrm{vert}(Q)$ be the representatives with coordinates in decreasing order:

$$p_1 \geq p_2 \geq \cdots \geq p_n$$
$$q_1 \geq q_2 \geq \cdots \geq q_n$$

Orbit Polytopes up to Normal Equivalence

Definition

The polytopes P and Q are normally equivalent if they have the same normal fan.

If P and Q are orbit polytopes in $\mathbb{R}^n$, let $p \in \mathrm{vert}(P)$ and $q \in \mathrm{vert}(Q)$ be the representatives with coordinates in decreasing order:

$$p_1 \geq p_2 \geq \cdots \geq p_n$$
$$q_1 \geq q_2 \geq \cdots \geq q_n$$

Normal equivalence asks which of the $\geq$ are actually $=$.

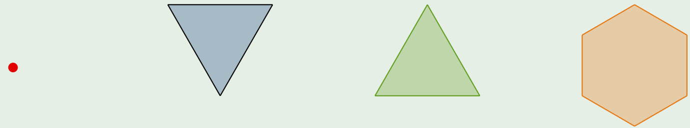
Orbit Polytopes up to Normal Equivalence

Normal equivalence classes of orbit polytopes are in bijection with integer compositions.

Examples

Orbit polytopes in $\mathbb{R}^3$:

$\mathcal{O}(0,0,0)$ $\mathcal{O}(1,1,0)$ $\mathcal{O}(1,0,0)$ $\mathcal{O}(2,1,0)$

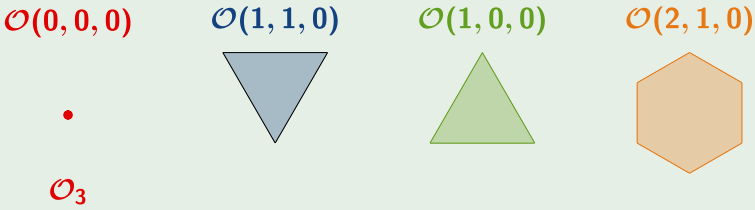


Orbit Polytopes up to Normal Equivalence

Normal equivalence classes of orbit polytopes are in bijection with integer compositions.

Examples

Orbit polytopes in $\mathbb{R}^3$:

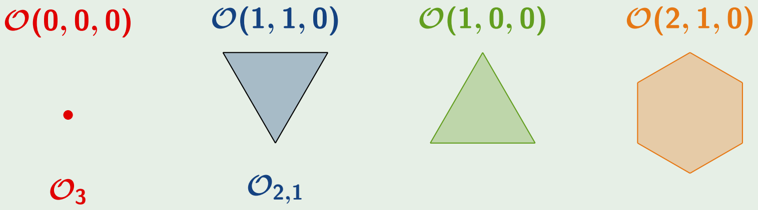


Orbit Polytopes up to Normal Equivalence

Normal equivalence classes of orbit polytopes are in bijection with integer compositions.

Examples

Orbit polytopes in $\mathbb{R}^3$:



Hopf Monoids

Orbit Polytopes

The Hopf Monoid of Orbit Polytopes

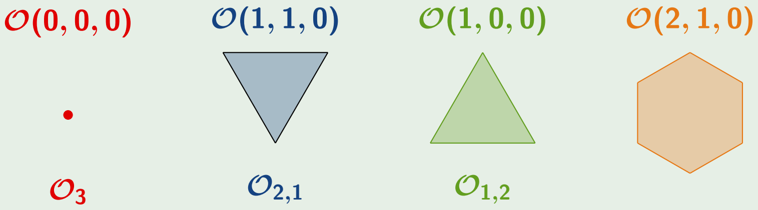
Character Group of Orbit Polytopes

Orbit Polytopes up to Normal Equivalence

Normal equivalence classes of orbit polytopes are in bijection with integer compositions.

Examples

Orbit polytopes in $\mathbb{R}^3$:



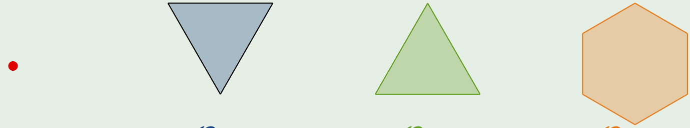
Orbit Polytopes up to Normal Equivalence

Normal equivalence classes of orbit polytopes are in bijection with integer compositions.

Examples

Orbit polytopes in $\mathbb{R}^3$:

$\mathcal{O}(0,0,0)$ $\mathcal{O}(1,1,0)$ $\mathcal{O}(1,0,0)$ $\mathcal{O}(2,1,0)$



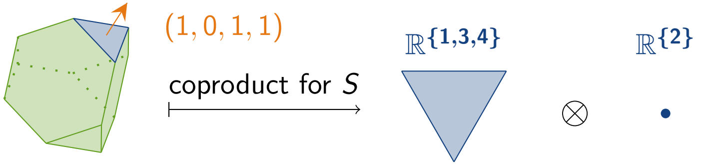
$\mathcal{O}_3$ $\mathcal{O}_{2,1}$ $\mathcal{O}_{1,2}$ $\mathcal{O}_{1,1,1}$

The Hopf Monoid of Orbit Polytopes

- Generated by orbit polytopes
- Product: standard product of polytopes
- Coproduct (for generalized permutahedra):
  (i) Start with any $S \subseteq [n]$
  (ii) Find maximal face in direction $e_S = \sum_{i \in S} e_i$
  (iii) (nontrivial!!) That face decomposes as a product of a generalized permutahedron in $\mathbb{R}^S$ and one in $\mathbb{R}^{[n]-S}$
- **Hopf structure on orbit polytopes induces Hopf structure on compositions**

Coproduct Example

$n = 4, \; S = \{1, 3, 4\}, \; e_S = (1, 0, 1, 1)$

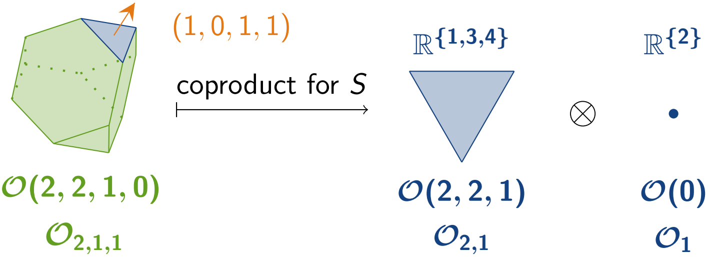


$\mathcal{O}(2, 2, 1, 0)$ $\quad$ $\mathcal{O}(2, 2, 1)$ $\quad$ $\mathcal{O}(0)$

$\mathcal{O}_{2,1,1}$ $\quad$ $\mathcal{O}_{2,1}$ $\quad$ $\mathcal{O}_{1}$

Coproduct Example

$n = 4, \;\; S = \{1, 3, 4\}, \;\; e_S = (1, 0, 1, 1)$



We are "chopping" the composition $(2, 1, 1)$:

$$(2, 1 \,\vdots\, 1) \mapsto (2, 1) \otimes (1)$$

Coproduct for Orbit Polytopes

Given a composition α, what are all the possible outcomes of taking a coproduct?

Coproduct for Orbit Polytopes

Given a composition α, what are all the possible outcomes of taking a coproduct?

Definition

The concatenation of compositions β and γ is

$$\beta \cdot \gamma = (\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_\ell).$$

The near-concatenation of β and γ is

$$\beta \odot \gamma = (\beta_1, \ldots, \beta_k + \gamma_1, \ldots, \gamma_\ell).$$

Coproduct for Orbit Polytopes

Given a composition α, what are all the possible outcomes of taking a coproduct?

Definition

The concatenation of compositions β and γ is

$$\beta \cdot \gamma = (\beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_\ell).$$

The near-concatenation of β and γ is

$$\beta \odot \gamma = (\beta_1, \ldots, \beta_k + \gamma_1, \ldots, \gamma_\ell).$$

Since we can either "chop" in between parts or in the middle of a part, we will obtain a pair of compositions β and γ such that either $\beta \cdot \gamma = \alpha$ or $\beta \odot \gamma = \alpha$.

Character Groups

Definition

The character group of a Hopf monoid is the collection of multiplicative functions from the Hopf monoid to a field $\mathbb{k}$, equipped with a convolution product $*$:

$$(\zeta * \psi)(x) = \sum_{S \subseteq [n]} \mathrm{mult}_{\mathbb{k}} \circ (\zeta \otimes \psi) \circ \mathrm{coproduct}_S(x)$$

The Character Group of Orbit Polytopes

Theorem [S. 2019]

The character group of the Hopf monoid of orbit polytopes is isomorphic to the collection of generating functions of the form

$$\sum_{\beta \in \mathrm{Comp}} \frac{c_\beta}{|\beta|!} R_\beta$$

under multiplication, where R_β is a collection of symbols indexed by compositions with the product

$$R_\beta R_\gamma = R_{\beta.\gamma} + R_{\beta \odot \gamma}$$

and $c_\beta \in \Bbbk$ with $c_\varnothing = 1$ and $c_{(n)} = c_{(1)}^n$ for all n.

Connection to *NSym*

- *NSym*: Hopf algebra of noncommutative symmetric functions
- One basis of *NSym* is the ribbon basis $\{R_\beta\}$ indexed by compositions, with $R_\beta R_\gamma = R_{\beta.\gamma} + R_{\beta\odot\gamma}$

Alternate formulation of the theorem [S. 2019]

The character group of the Hopf monoid of orbit polytopes is isomorphic to a subgroup of the group of invertible elements in the completion of *NSym*.

Thank you!

The Hopf Monoid of Orbit Polytopes

Mariel Supina

Hopf Monoids

Orbit Polytopes

The Hopf Monoid of Orbit Polytopes

Character Group of Orbit Polytopes

M. Supina, *The Hopf monoid of orbit polytopes*, April 2019, arXiv:1904.08437.