Characteristic elements for real hyperplane arrangements

Jose Bastidas

Cornell University

Joint work with M. Aguiar and S. Mahajan

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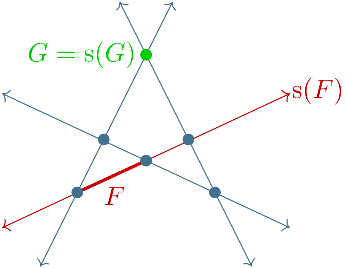
Hyperplane arrangements

Let $\mathcal{A}$ be a hyperplane arrangement: a finite collection of affine hyperplanes in a finite-dimensional real vector space V.

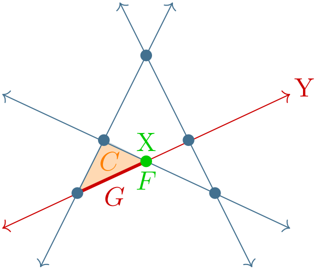
- The set of flats (non-empty intersections of hyperplanes) of $\mathcal{A}$ is denoted by $\Pi[\mathcal{A}]$.
- The hyperplanes in $\mathcal{A}$ split V into a collection of convex sets called faces. The set of faces of $\mathcal{A}$ is denoted by $\Sigma[\mathcal{A}]$.
- The support $\mathrm{s}(F)$ of a face F is the minimal flat containing it.



A hyperplane arrangement with
4 hyperplanes
11 flats
33 faces

Flats and faces

- $\Pi[\mathcal{A}]$ and $\Sigma[\mathcal{A}]$ are ordered by inclusion.
- $\top = V$ is the maximum of $\Pi[\mathcal{A}]$.
- Maximal elements of $\Sigma[\mathcal{A}]$ are called chambers.
- The support map $\mathrm{s} : \Sigma[\mathcal{A}] \to \Pi[\mathcal{A}]$ is order preserving.



$F \leq G \leq C$ in $\Sigma[\mathcal{A}]$

and

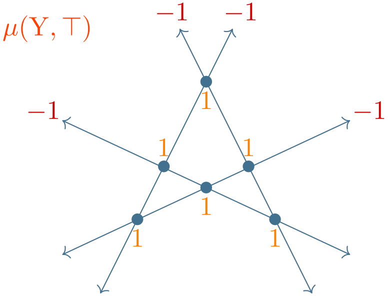
$\mathrm{X} \leq \mathrm{Y} \leq \top$ in $\Pi[\mathcal{A}]$

The characteristic polynomial

The characteristic polynomial of $\mathcal{A}$ is

$$\chi(\mathcal{A}, t) = \sum_{Y} \mu(Y, \top) t^{\dim Y}$$

where μ is the Möbius function of $\Pi[\mathcal{A}]$. The sum is over all flats of $\mathcal{A}$.

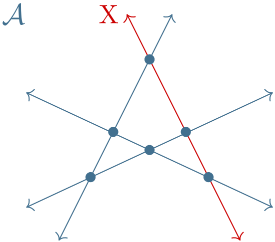


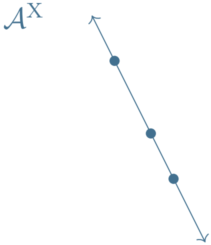
$$\chi(\mathcal{A}, t) = t^2 - 4t + 6$$

Arrangement under a flat

The **arrangement under a flat** X is the arrangement in ambient space X consisting of the following hyperplanes

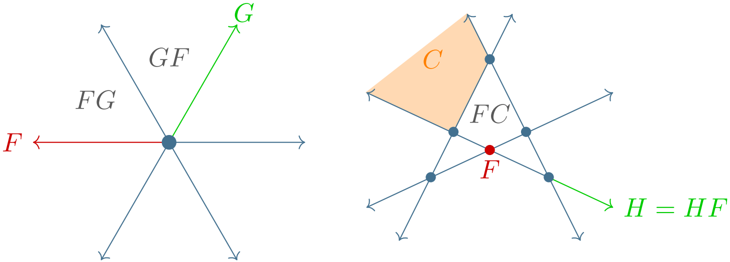
$$\mathcal{A}^{\mathrm{X}} = \{\mathrm{Y} \in \Pi[\mathcal{A}] \mid \mathrm{Y} \lessdot \mathrm{X}\}.$$





Tits semigroup

The set of faces $\Sigma[\mathcal{A}]$ forms a semigroup.
The product FG of a face F and a face G is the first face you encounter after moving a small positive distance from an interior point of F to an interior point of G.



- The product is not commutative.
- F is always a face of FG.
- $\mathrm{s}(FG) = \mathrm{s}(F) \vee \mathrm{s}(G)$

Characteristic elements

Fix a field $\Bbbk$ and consider the algebra $\Bbbk\Sigma[\mathcal{A}]$. A general element of $\Bbbk\Sigma[\mathcal{A}]$ is of the form

$$w = \sum_F w^F F$$

for some scalars $w^F \in \Bbbk$.

Definition/Lemma

Let $t \in \Bbbk$. An element $w \in \Bbbk\Sigma[\mathcal{A}]$ is characteristic of parameter t if for each flat X

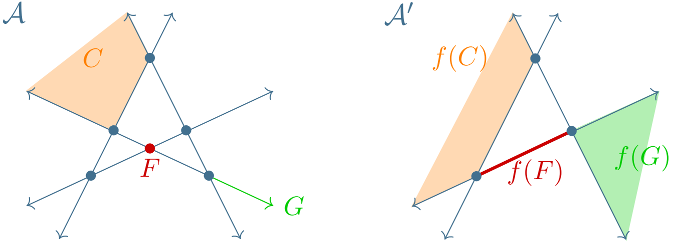
$$\chi_{\mathrm{X}}(w) := \sum_{\mathrm{s}(F) \leq \mathrm{X}} w^F = t^{\dim \mathrm{X}}.$$

Equivalently, if for each flat X

$$\sum_{\mathrm{s}(F) = \mathrm{X}} w^F = \chi(\mathcal{A}^{\mathrm{X}}, t).$$

Application: The fundamental recursion

Let $\mathcal{A}'$ be a subarrangement of $\mathcal{A}$. For each face F of $\mathcal{A}$, let $f(F)$ be the minimal face of $\mathcal{A}'$ containing it.



The map $f : \Sigma[\mathcal{A}] \to \Sigma[\mathcal{A}']$ is a morphism of semigroups, so it extends to a morphism of algebras $f : \Bbbk\Sigma[\mathcal{A}] \to \Bbbk\Sigma[\mathcal{A}]$

Lemma

If $w \in \Bbbk\Sigma[\mathcal{A}]$ is characteristic, so is $f(w) \in \Bbbk\Sigma[\mathcal{A}']$ (of the same parameter).

Application: The fundamental recursion

The face $f(F)$ is a chamber of $\mathcal{A}'$ if and only if F is not contained in any hyperplane of $\mathcal{A}'$.

$$\chi(\mathcal{A}', t) = \sum_{C'} f(w)^{C'} = \sum_{C'} \Big(\sum_{f(F)=C'} w^F \Big) = \sum_{\mathrm{X}} \chi(\mathcal{A}^{\mathrm{X}}, t)$$

where the last sum run over all flats X of $\mathcal{A}$ not contained in any hyperplane of $\mathcal{A}'$.

If $\mathcal{A}' = \mathcal{A} \setminus \{\mathrm{H}\}$ for some hyperplane $\mathrm{H} \in \mathcal{A}$, the formula above is equivalent to

$$\chi(\mathcal{A}, t) = \chi(\mathcal{A} \setminus \{\mathrm{H}\}, t) - \chi(\mathcal{A}^{\mathrm{H}}, t)$$

Application: A formula by Kung

Lemma

If u and v are characteristic elements of parameters s and t, then uv is characteristic of parameter st.

Let u and v be characteristic elements of parameters s and t.

$$\chi(\mathcal{A}, st) = \sum_{C} (uv)^C = \sum_{\mathrm{s}(FG)=\top} u^F v^G = \sum_{\mathrm{X} \vee \mathrm{Y} = \top} \chi(\mathcal{A}^{\mathrm{X}}, s)\chi(\mathcal{A}^{\mathrm{Y}}, t).$$

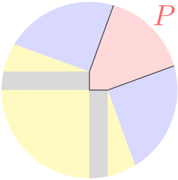
Note that $\mathrm{X} \vee \mathrm{Y} = \top$ if and only if Y is not contained in any hyperplane of $\mathcal{A}_{\mathrm{X}} := \{\mathrm{H} \in \mathcal{A} \mid \mathrm{X} \leq \mathrm{H}\}$.
Using the formula from the previous slide we get:

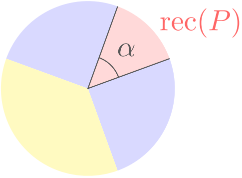
$$\chi(\mathcal{A}, st) = \sum_{\mathrm{X}} \chi(\mathcal{A}^{\mathrm{X}}, s)\chi(\mathcal{A}_{\mathrm{X}}, t).$$

Intrinsic volumes

Let P be a (convex) polyhedron in $\mathbb{R}^n$ (like the faces of an arrangement).
The k-dimensional intrinsic volume of P is

$$v_k(P) = \begin{array}{c}\text{proportion of } \mathbb{R}^n \text{ that projects to the interior} \\ \text{of a } k\text{-dimensional face of } P \text{ under} \\ \text{the nearest point projection } \pi_P : \mathbb{R}^n \to P\end{array}$$





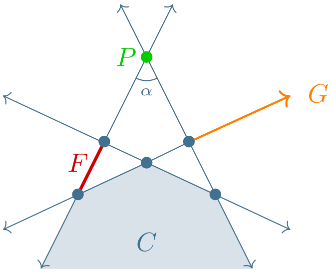
$$v_2(P) = \frac{\alpha}{2\pi}$$
$$v_1(P) = \frac{1}{2}$$
$$v_0(P) = \frac{1}{2} - \frac{\alpha}{2\pi}$$

Intrinsic elements

Definition

The intrinsic element of parameter t is

$$\nu_t = \sum_F \Big(\sum_k (-1)^{\dim(F)-k} v_k(F) t^k \Big) F.$$



$$\nu_t{}^P = 1$$
$$\nu_t{}^F = -1$$
$$\nu_t{}^G = \tfrac{1}{2}t - \tfrac{1}{2}$$
$$\nu_t{}^C = \tfrac{\alpha}{2\pi}t^2 - \tfrac{1}{2}t + \left(\tfrac{1}{2} - \tfrac{\alpha}{2\pi}\right)$$

Intrinsic elements

Theorem (ABM19+)

For all $t \in \Bbbk$, ν_t is a characteristic element of parameter t.

Corollary (Klivans, Swartz)

The coefficient of t^k in the characteristic polynomial of $\mathcal{A}$ is

$$(-1)^{n-k} \sum_{C} v_k(C)$$

where n is the dimension of the ambient space. The sum is taken over all chambers of $\mathcal{A}$.

Intrinsic elements

Theorem (ABM19+)

For any parameters $s, t \in \Bbbk$, we have

$$\nu_s \nu_t = \nu_{st}.$$

Remark

If $\mathcal{A}$ is linear, the coefficients of opposite faces in ν_t are equal.

Problem:

Describe other families $\{w_t\}_t$ of characteristic elements satisfying these two properties.

- For the braid arrangement, such a family that can be defined by *counting lattice points*.
- Any such family must satisfy $w_1 = \upsilon = \nu_1$ and $w_{-1} = \nu_{-1}$.

Thank you!

Application: Zaslavsky's formula

Theorem (AM17)

The algebra $\Bbbk\Sigma[\mathcal{A}]$ is unital (even when $\Sigma[\mathcal{A}]$ is not). The unit element is

$$\upsilon = \sum_F (-1)^{\mathrm{rk}(F)} F$$

with F running over the set of essentially bounded *faces of $\mathcal{A}$.*

The unit υ is a characteristic element of parameter one.

$$\chi(\mathcal{A}, 1) = \sum_C \upsilon^C$$

Therefore

$$(-1)^{\mathrm{rk}(\mathcal{A})}\chi(\mathcal{A}, 1) = (-1)^{\mathrm{rk}(\mathcal{A})} \sum_{\mathrm{X}} \mu(\mathrm{X}, \top)$$

counts the number of bounded chambers of $\mathcal{A}$.

Example: Adams element

Let $\mathcal{A}$ be the **braid arrangement** in $\mathbb{R}^n$. It consists of the diagonal hyperplanes $x_i = x_j$ for $1 \leq i < j \leq n$.

The **Adams element** of parameter t is defined by

$$\alpha_t = \sum_F \binom{t}{\dim(F)} F$$

If t is a positive integer,

- $\binom{t}{\dim(F)}$ counts the number of points in the **interior** of F with coordinates in $[t] = \{1, 2, \ldots, t\}$.
- The number of points in $[t]^n \cap \mathrm{X}$ is $t^{\dim(\mathrm{X})}$.

Therefore, α_t is a characteristic element of parameter t.

Corollary

The characteristic polynomial of $\mathcal{A}$ is

$$\sum_C {\alpha_t}^C = n! \binom{t}{n} = t(t-1)\ldots(t-n+1)$$